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A Topological Theory of the Electromagnetic Field

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Abstract. It is shown that Maxwell equations in vacuum derive from an underlying topological structure given by a scalar field ϕ which represents a map $S^3 \times R \to S^2$ and determines the electromagnetic field through a certain transformation, which also linearizes the highly nonlinear field equations to the Maxwell equations. As a consequence, Maxwell equations in vacuum have topological solutions, characterized by a Hopf index equal to the linking number of any pair of magnetic lines. This allows the classification of the electromagnetic fields into homotopy classes, labeled by the value of the helicity. Although the model makes use of only c-number fields, the helicity always verifies $\int \mathbf{A} \cdot \mathbf{B} \, d^3\mathbf{r} = n\alpha$, n being an integer and α an action constant, which necessarily appears in the theory, because of reasons of dimensionality.

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1. Introduction

Topology will play a very important role in future field theory. Since 1931, when Dirac proposed his beautiful idea of the monopole, topological models have a growing place in physics. For the purpose of introducing this Letter, we could summarize this line of work by quoting the sine–Gordon equation, the 't Hooft–Polyakov monopole, the Skyrme and Faddeev models, the Bohm–Aharonov effect, Berry's phase, or Chern–Simons terms [1–10].

This Letter proposes a model of an electromagnetic field in which the magnetic helicity $\int \mathbf{A} \cdot \mathbf{B} \, d^3\mathbf{r}$ is a topological constant of the motion, which allows the classification of the possible fields into homotopy classes, as it is equal to the linking number of any pair of magnetic lines. The treatment is classical throughout in the sense that all the fields are c-numbers and no second quantization is performed. The case of q-numbers is surely much more complex.

2. Electromagnetic Field Model with Hopf Index

Let $\phi(\mathbf{r}, \mathbf{t})$ and $\theta(\mathbf{r}, \mathbf{t})$ be two complex scalar fields representing maps $R^3 \times R \to C$. By identifying $R^3 \cap \{\infty\}$ with S^3 and $C \cap \{\infty\}$ with S^2, via stereographic projection, ϕ and θ can be understood as maps $S^3 \times R \to S^2$. We then define the antisymmetric tensors $F_{\mu\nu}$, $G_{\mu\nu}$ to be equal to

$$F_{\mu\nu} = f_{\mu\nu}(\phi) = \frac{\sqrt{a}}{2\pi i} \frac{\partial_\mu \phi^* \, \partial_\nu \phi - \partial_\nu \phi^* \, \partial_\mu \phi}{(1 + \phi^* \phi)^2}, \tag{1}$$

$$G_{\mu\nu} = f_{\mu\nu}(\theta) = \frac{\sqrt{a}}{2\pi i} \frac{\partial_\mu \theta^* \, \partial_\nu \theta - \partial_\nu \theta^* \, \partial_\mu \theta}{(1 + \theta^* \theta)^2}, \tag{2}$$

where a is an action constant, introduced so that $F_{\mu\nu}$ and $G_{\mu\nu}$ will have suitable dimensions for electromagnetic fields, and prescribe that G_{ij} be the dual of F_{ij} or, equivalently,

$$G_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}, \qquad F_{\mu\nu} = -\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta} G^{\alpha\beta}, \tag{3}$$

where we take $\epsilon^{0123} = +1$. To fulfill this requirement, ϕ is a scalar and θ a pseudoscalar. This allows the defining of the magnetic and electric fields $\mathbf{B}$ and $\mathbf{E}$ as

$$F_{0i} = E_i, \quad F_{ij} = -\epsilon_{ijk} B_k; \qquad G_{0i} = B_i, \qquad G_{ij} = \epsilon_{ijk} E_k. \tag{4}$$

After that, we take the Lagrangian density

$$L = -\tfrac{1}{8}(F_{\mu\nu} F^{\mu\nu} + G_{\mu\nu} G^{\mu\nu}), \tag{5}$$

We then impose the duality condition or constraint

$$M_{\sigma\beta} = G_{\alpha\beta} - \tfrac{1}{2}\epsilon_{\alpha\beta\mu\nu} F^{\mu\nu} = 0. \tag{6}$$

Following the method of Lagrange multipliers, instead of (5), we take the modified Lagrangian density

$$L' = L + \mu^{\alpha\beta} M_{\alpha\beta}, \tag{7}$$

where the multipliers are the component of the constant tensor $\mu^{\alpha\beta}$. A simple calculation shows that constraint (6) does not contribute to the Euler–Lagrange equations, which happen to be

$$\partial_\alpha F^{\alpha\beta} \, \partial_\beta \phi = 0, \qquad \partial_\alpha F^{\alpha\beta} \, \partial_\beta \phi^* = 0, \tag{8a,b}$$

$$\partial_\alpha G^{\alpha\beta} \, \partial_\beta \theta = 0, \qquad \partial_\alpha G^{\alpha\beta} \, \partial_\beta \theta^* = 0, \tag{9a,b}$$

and this means that, if the Cauchy data $(\phi, \partial_0 \phi, \theta, \partial_0 \theta)$ at $t = 0$ verify the constraint (6), it will be maintained for all $t > 0$. Surprisingly, it follows that both $F_{\alpha\beta}$ and $G_{\alpha\beta}$ verify Maxwell equations in vacuum. In fact, definitions (1) and (2) imply that

$$\epsilon^{\alpha\beta\gamma\delta} \, \partial_\beta F_{\gamma\delta} = 0, \qquad \epsilon^{\alpha\beta\gamma\delta} \, \partial_\beta G_{\gamma\delta} = 0, \tag{10a,b}$$

independently of the functions ϕ and θ, this being precisely the first pair of Maxwell equations for the two tensors. On the other hand, as $F_{\alpha\beta}$ and $G_{\alpha\beta}$ are dual to each other, it follows from (3) and (10) that

$$\partial_\alpha F^{\alpha\beta} = 0, \qquad \partial_\alpha G^{\alpha\beta} = 0, \quad \beta = 0, 1, 2, 3, \tag{11a,b}$$

which is the second Maxwell pair for the two tensors. In other words, *if ϕ and θ obey the Euler–Lagrange equations (8), (9), then $F_{\alpha\beta}$ and $G_{\alpha\beta}$, defined by (1), (2), verify the Maxwell ones and are, therefore, electromagnetic fields of the standard theory.* The reason is that Maxwell equations in vacuum have the property that, if two dual tensors verify the first pair, they also verify the second one (i.e. the two pairs are dual to each other).

The following notation will be used. A standard electromagnetic field is any solution of Maxwell equations; an admissible electromagnetic field is one which can be deduced from a scalar ϕ through (1). In this last case, let $f_{\mu\nu}(\phi)$ be the electromagnetic tensor $F_{\mu\nu}$. The electric and magnetic vectors of ϕ, $\mathbf{E}(\phi)$ and $\mathbf{B}(\phi)$, respectively, are

$$E_i(\phi) = f_{0i}(\phi), \qquad B_i(\phi) = -\tfrac{1}{2}\epsilon_{ijk} f_{jk}(\phi). \tag{12}$$

With this notation, the duality constraint is written

$$\mathbf{E}(\phi) = -\mathbf{B}(\theta), \qquad \mathbf{B}(\phi) = \mathbf{E}(\theta). \tag{13}$$

It is necessary to characterize the Cauchy data $\{\phi(\mathbf{r}, 0), \partial_t\phi(\mathbf{r}, 0), \theta(\mathbf{r}, 0), \partial_t\theta(\mathbf{r}, 0)\}$. As was shown before, if they verify condition (3) at $t = 0$, they also satisfy it for all $t > 0$. In this case, the Cauchy data and the corresponding solution of Maxwell equations are admissible. The problem then arises whether there exist admissible data, but that seems guaranteed at first sight, since (3) is a set of six real PDEs for eight real functions. That it is indeed the case and admissible data exist, is clear from the following considerations.

From the two facts (a) $\mathbf{E}(\phi)$ and $\mathbf{B}(\phi)$ are mutually orthogonal (because of (1)) and (b) $\mathbf{B}(\phi)$ is tangent to the curves $\phi = \text{const}$ and $\mathbf{B}(\theta)$ is tangent to $\theta = \text{const}$, it follows that these two sets of curves must be orthogonal. Let $\phi(\mathbf{r}, 0)$ be any complex function with the only condition that the 1-forms $\mathrm{d}\phi$ and $\mathrm{d}\phi^*$ in R^3 are linearly independent. The previous condition on θ can be written as

$$(\nabla\phi^* \times \nabla\phi) \cdot (\nabla\theta^* \times \nabla\theta) = 0, \tag{14}$$

which, given ϕ, is a complex PDE for the complex function θ; we will admit that it has solutions (later we will make use of an explicit example). This gives $\phi(\mathbf{r}, 0)$ and $\theta(\mathbf{r}, 0)$. The time derivatives $\partial_t\phi(\mathbf{r}, 0)$ and $\partial_t\theta(\mathbf{r}, 0)$ are then fixed by condition (13). For instance, $\mathbf{B}(\theta)$ is a linear combination of $\nabla\phi^*$ and $\nabla\phi$,

$$\mathbf{B}(\theta) = b\, \nabla\phi^* + b^*\, \nabla\phi. \tag{15}$$

The function $b(\mathbf{r}, 0)$ can be determined from $\phi(\mathbf{r}, 0)$ and $\theta(\mathbf{r}, 0)$ and, according to (1),

$$\mathbf{E}(\phi) = \frac{1}{2\pi i} \frac{\partial_0\phi^*\, \nabla\phi - \partial_0\phi \nabla\phi^*}{(1 + \phi^*\phi)^2} = -\mathbf{B}(\theta), \tag{16}$$

the value of $\partial_0\phi$ can be computed from (15) and (16). To obtain $\partial_0\theta$, we would proceed in an analogous way. Consequently, there is no difficulty with the Cauchy problem, the system having two degrees of freedom with a differential constraint.

Up to now, we have used a pair of fields (ϕ, θ), but it is easy to understand that θ is no more than a convenience which can be dispensed with. In fact, one can forget about θ and use only the scalar ϕ, taking

$$L = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} = f_{\mu\nu}(\phi), \tag{17}$$

as Lagrangian density and accepting only Cauchy data $[\phi(\mathbf{r}, 0), \partial_0\phi(\mathbf{r}, 0)]$ for which there exists an auxiliary function θ verifying (3) and (14). From this point of view, the electromagnetic field would be a scalar. From now on, the θ field will be considered only as an auxiliary function. The basic field equations of the model thus take the form

$$\partial^\mu F_{\mu\nu} = \sqrt{a}\,\partial^\mu\left[\frac{1}{2\pi i}\frac{\partial_\mu\phi^*\,\partial_\nu\phi - \partial_\nu\phi^*\,\partial_\mu\phi}{(1+\phi^*(\phi)^2}\right] = 0. \tag{18}$$

and are transformed into Maxwell equations through (1).

We may state these results as follows:

(i) In a theory of the fields ϕ and θ based on the Lagrangian (5) with the constraint (6), the tensors $F_{\alpha\beta}$ and $G_{\alpha\beta}$ defined by (1) and (2) obey Maxwell equations. This means that the standard electromagnetic theory can be derived from an underlying structure.

(ii) Formulae (1) and (2) can be understood as defining a transformation

$$T\colon \phi \to F_{\mu\nu} = f_{\mu\nu}(\phi), \qquad \theta \to G_{\mu\nu} = f_{\mu\nu}(\theta) \tag{19}$$

which transforms the highly nonlinear Euler–Lagrange equations in ϕ (8), (9) into Maxwell equations. We can thus say that (8) and (9) are C-integrable, at least in a weak form [11], since they can be linearized by change of variables.

(iii) The transformation T is not invertible, because there are solutions of Maxwell equations $H_{\mu\nu}$, such that $T^{-1}(H_{\mu\nu})$ is not defined, that is to say, that a scalar field ϕ such that $H_{\mu\nu} = f_{\mu\nu}(\phi)$ does not exist.

These solutions of Maxwell equations are not included in this theory. As we will see in Section 4, this exclusion is only important for nonweak fields.

The use of the spheres S^3 and S^2 may remind us of Chern–Simons terms. But, as S^3 represents the physical space R^3 via stereographic projection and S^2, identified with the complex plane, is the space where the field takes values, this model does not really make use of these kind of terms (for a review of Chern–Simons terms, see Jackiw [9–10]). Flato and Fronsdal proposed a very interesting scheme to make light out of simpler objects [12]. Although, this aim is shared by the present work, the two approaches seem to be different.

As will be shown in the next section, it is not possible to distinguish between this model and the Maxwell one if the fields are weak. However, every ϕ solution defines at any time t a map $S^3 \to S^2$ which has a topological charge, obviously independent on t.

3. The Topological Charge

Since the field ϕ defines a map $S^3 \to S^2$ at any time, the corresponding Hopf index [13–19], expressed as

$$n = \frac{1}{a}\int_{R^3} \mathbf{A}\cdot\mathbf{B}\,d^3\mathbf{r}, \tag{20}$$

where curl $\mathbf{A} = \mathbf{B}$, is a topological constant of the motion. Its value is an integer equal to the linking number of any pair of magnetic lines. This allows the classification of the solutions in homotopy classes.

The quantity $\int \mathbf{A} \cdot \mathbf{B}\, d^3\mathbf{r}$, known in plasma physics as magnetic helicity [20–21], is used in the study of tokamaks and is quite akin to the helicity used in fluid dynamics, where the roles of **B** and **A** are played by the vorticity and the velocity [22].

4. Maxwell Theory is the Weak Field Limit of This Model

Although our $F_{\mu\nu}$ verifies Maxwell equations, this model is not completely equivalent to standard theory. However, we shall prove in this section that, in the weak field case, this theory is equivalent to Maxwell plus the condition that the electric and magnetic vectors be orthogonal, $\mathbf{E} \cdot \mathbf{B} = 0$, in the sense that all the standard electromagnetic fields which are weak and verify this requisite are arbitrarily close to others which are admissible. In order to do that, we must introduce some definitions and fix the notation.

If $\phi^*\phi \ll 1$ so that the denominator in (1) can be taken as being equal to one, then

$$F_{\mu\nu} = \tilde{f}_{\mu\nu}(\phi) = \frac{\sqrt{a}}{2\pi i}(\partial_\mu\phi^*\,\partial_\nu\phi - \partial_\nu\phi^*\,\partial_\mu\phi). \tag{21}$$

This equation defines the meaning of $\tilde{f}_{\mu\nu}$. As a consequence, if ϕ is a solution, $\lambda\phi$ is another one or, otherwise stated, if $(\mathbf{E}, \mathbf{B})$ is a solution, $(\lambda^*\lambda\mathbf{E}, \lambda^*\lambda\mathbf{B})$ is also a solution.

The norms of a scalar $\phi(\mathbf{r}, \mathbf{t})$ and a tensor $F_{\mu\nu}(\mathbf{r}, t) = [\mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, r)]$ in a domain Σ are defined as

$$\|\phi\|_\Sigma = \sup\{|\phi|, |\partial_\alpha\phi|\} \quad (\alpha = 0, 1, 2, 3, (\mathbf{r}, t) \in \Sigma), \tag{22}$$

$$\|F\|_\Sigma = \sup\{|\mathbf{B}_i|, |\mathbf{E}_i|\} \quad (i = 1, 2, 3, (\mathbf{r}, t) \in \Sigma). \tag{23}$$

In the following, Ω will refer to a domain in spacetime R^* and D to a domain in three-space R^3. The subscripts Ω or D will be omitted when there is no risk of confusion. The fields ϕ or $F_{\mu\nu}$ will be said to be weak if their norms are small.

We now proceed to prove that, although not all electromagnetic fields $F_{\mu\nu}$ can be deduced from a scalar according to (1), all can be derived from a ϕ by means of (21). This is the content of Propositions 1 and 2.

PROPOSITION 1. *Let a divergenceless vector field* **B** *be given in* R^3. *There then exists a complex function* $\phi = \alpha + i\beta$, *such that its limit when* $r \to \infty$ *does not depend on the direction and such that, by defining the antisymmetric tensor* F_{jk} *through* (21), *the following equality holds*

$$B_i = -\tfrac{1}{2}\epsilon_{ijk}F_{jk} = -\frac{\sqrt{a}}{\pi}(\nabla\alpha \times \nabla\beta)_i, \tag{24}$$

Moreover, if **B** *is small enough,* ϕ *can be made as small as desired.*

The fact that the vorticity ω can be written as the vector product of two gradients is well known in fluid mechanics where α and β, called Clebsch variables (see, for instance, [23] or [24]), are used to put the fluid equations into Hamiltonian form. Since the proof is based in the divergenceless character of ω, it also applies to the case of the magnetic field and, for brevity, will not be given here. Obviously, α and β are not uniquely determined. For instance, we could take $(\alpha + \sigma\beta, \beta)$ instead of (α, β), σ being a constant number.

We now assume that $\mathbf{B}$ is bounded in a certain domain D of R^3. If $\mathbf{B}$ is changed to $\sigma^2\mathbf{B}$, it suffices changing α, β to $\sigma\alpha$, $\sigma\beta$, from which it follows that if $\mathbf{B}$ is small enough in D, $\phi^*\phi = \alpha^2 + \beta^2$ is also small in D.

PROPOSITION 2. *Let us assume the hypothesis of Proposition* 1 *and let a field* $\mathbf{E}$ *orthogonal to* $\mathbf{B}$ *be given in* R^3. *Then there exists a complex function* $\partial_0\phi = \partial_0\alpha + i\,\partial_0\beta$ *such that*

$$E_i = \frac{\sqrt{a}}{\pi}(\partial_0\alpha\,\partial_i\beta - \partial_0\beta\,\partial_i\alpha). \tag{25}$$

The *proof* is simpie. As $\mathbf{E}$ is orthogonal to $\mathbf{B}$, it is a linear combination of the gradients $\nabla\alpha$, $\nabla\beta$, so that it is always possible to find functions $\partial_0\alpha$ and $\partial_0\beta$, verifying (25). Moreover, they are small in the domain D if $\mathbf{E}$ and $\mathbf{B}$ are small enough in D.

If $F_{\mu\nu}$ is a standard electromagnetic field which is weak enough, there is an admissible field arbitrarily close to it. This is proved in a precise form in the following proposition.

PROPOSITION 3. *Let* $F_{\mu\nu}(\mathbf{r}, t)$ *be a solution of Maxwell equations corresponding to the Cauchy data* $F_{\mu\nu}(\mathbf{r}, 0)$ *which verify* $\mathbf{E}\cdot\mathbf{B} = 0$ *at* $t = 0$. *Then, if* D *is a domain in* R^3, *for any* $\epsilon > 0$, *there is an* $\eta = \eta(\epsilon, D)$ *such that if* $\|F_{\mu\nu}(\mathbf{r}, 0)\|_D < \eta$, *there exist a scalar* ϕ *and an admissible electromagnetic field* $F'_{\mu\nu} = f_{\mu\nu}(\phi)$, *such that in a four-dimensional domain* Ω, $R^4 \supset \Omega \supset D$, $F_{\mu\nu}$ *and* $F'_{\mu\nu}$ *are closer than* ϵ *in the sense of the following inequality*

$$\left|\frac{F'_{\mu\nu} - F_{\mu\nu}}{F_{\mu\nu}}\right| < \epsilon \quad (\mu, \nu = 0, 1, 2, 3). \tag{26}$$

Proof. According to Proposition 1, two functions $\psi(\mathbf{r}, 0)$, $\xi(\mathbf{r}, 0)$ exist such that

$$B_i = -\tfrac{1}{2}\epsilon_{ijk}\tilde{f}_{jk}(\psi), \qquad E_i = -\tfrac{1}{2}\epsilon_{ijk}\tilde{f}_{jk}(\xi)$$

and, following Proposition 2, derivatives $\partial_0\psi(\mathbf{r}, 0)$, $\partial_0\xi(\mathbf{r}, 0)$ also exist such that, at $t = 0$,

$$F_{\mu\nu} = \tilde{f}_{\mu\nu}(\psi). \tag{27}$$

Moreover, by taking $F_{\mu\nu}$ small enough, we can make $\|\psi\| < \epsilon'$, $\|\xi\| < \epsilon'$, in any domain in which $F_{\mu\nu}$ is bounded, for any ϵ'.

As the curves $\psi(\mathbf{r}, 0) = \text{const}$ and $\xi(\mathbf{r}, 0) = \text{const}$ are orthogonal, we can define admissible Cauchy data as indicated in Section 2 with $\phi(\mathbf{r}, 0) = \psi(\mathbf{r}, 0)$,

$\theta(\mathbf{r}, 0) = \xi(\mathbf{r}, 0)$ and the correlative values of $\partial_0\phi(\mathbf{r}, 0)$ and $\partial_0\theta(\mathbf{r}, 0)$. Let the corresponding electromagnetic field be denoted as $F'_{\mu\nu}$. At $t = 0$, the following is true

$$B_i = -\tfrac{1}{2}\epsilon_{ijk}\bar{f}_{jk}(\phi), \qquad B'_i = -\tfrac{1}{2}\epsilon_{ijk}f_{jk}(\phi), \tag{28}$$

$$E_i = -\tfrac{1}{2}\epsilon_{ijk}\bar{f}_{jk}(\theta), \qquad E'_i = -\tfrac{1}{2}\epsilon_{ijk}f_{jk}(\theta), \tag{29}$$

From this and (1) and (28), it follows that

$$B'_i = B_i[1 - 2\phi^*\phi + O((\phi^*\phi)^2)], \qquad B'_i = B_i[1 - 2\phi^*\phi + O((\phi^*\phi)^2)] \tag{30}$$

and, as if $t = 0$, $\phi^*\phi \leqslant \epsilon'$, $\theta^*\theta \leqslant \epsilon'$, inequality (27) follows, after taking $\epsilon' < \epsilon/2$.

Hence, there are two sets of Cauchy data, $F_{\mu\nu}(\mathbf{r}, 0)$ and $\mathbf{F}'_{\mu\nu}(\mathbf{r}, 0)$, which are arbitrarily close. The properties of Maxwell equations guarantee that this proximity is maintained in a four-dimensional domain $\Omega \supset D$.

The consequence is important. Any weak-enough standard electromagnetic field with $\mathbf{E} \cdot \mathbf{B} = 0$ at $t = 0$ is arbitrarily close to a set of Cauchy data which are admissible in this theory. The physical interpretation of this property is that, for weak electromagnetic fields, this theory reduces to Maxwell, plus the restriction $\mathbf{E} \cdot \mathbf{B} = 0$, at zero order in the norm of $F_{\mu\nu}$. For, taking as Cauchy data the electromagnetic tensor $F_{\mu\nu}$ at time t, there always exist functions ϕ and $\partial_0\phi$ from which they can be deduced according to (1), at zero order in $\phi^*\phi$ (if one can neglect this quantity in the denominator). However, it must be stressed that this is not the case for strong fields, since, although a function ϕ exists which verifies (21), another one does not necessarily exist for which (1) is valid, so that large standard electromagnetic fields may not be included in this model. It is clear that if ϕ is weak only in a certain domain of spacetime, the model would only reduce there to the standard one.

If the theory is observed from the point of view of the vectors $\mathbf{E}$ and $\mathbf{B}$, it seems linear, although it is not so by any means. It can be said, therefore, that the nonlinearity is hidden. In spite of that, it has two important consequences. Not all the strong field solutions of Maxwell equations are admissible and there is a topological constant of the motion, given by the Hopf index (20).

5. Standard Electromagnetic Fields which are Hopf Knots

The field equations (18) are very difficult to solve. Perhaps their most interesting problem is to know what the basic solutions with a unit Hopf index look like. As they represent elementary structures, nonhomotopic to the zero solution, from which solutions with higher values of the index can be constructed, it could be tempting to interpret them as photons. However, this turns out not to be possible, although they can be appropriately called quasiphotons.

Hopf himself showed that there are $S^3 \to S^2$ maps with nonzero indexes by proposing the following example, written here in R^3 Cartesian coordinates [13, 14]

$$\phi_H(x, y, z) = \frac{2(x + iy)}{2z + i(r^2 - 1)}, \tag{31}$$

where $r^2 = x^2 + y^2 + z^2$. It is easy to verify that this map, called the Hopf map or knot, has unit Hopf index; its **B** vector has a vortex along the circle $x^2 + y^2 = 1$, $z = 0$, which is the inverse image of $\phi = \infty$.

Corresponding to this Hopf map, in this theory, there exist a solution ϕ and a corresponding electromagnetic field $F_{\mu\nu}$. Let us consider the initial Cauchy data

$$\phi(\mathbf{r}, 0) = \phi_H(\kappa x, \kappa y, \kappa z), \qquad \theta(\mathbf{r}, 0) = [\phi_H(\kappa y, \kappa z, \kappa x)]^*, \tag{32}$$

where ϕ_H is the Hopf map (31), the asterisk means complex conjugation, and κ is an inverse length parameter giving the extension of the wave. At $t = 0$, ϕ and θ are Hopf knots around the axis z and x, respectively. In other words, a magnetic vortex in the circle $x^2 + y^2 = 1$, $z = 0$ and an electric one in $y^2 + z^2 = 1$, $x = 0$, corresponding to $\phi = \infty$ and $\theta = \infty$, respectively. It is simple to see that ϕ and θ verify condition (14) and are, therefore, admissible as Cauchy data. The corresponding electromagnetic field is a solution of Maxwell equations with the Hopf index equal to one, which means that any pair of magnetic lines is linked once (for symmetry, the same happens to any pair of electric lines). To generate solutions with the Hopf index equal to n, it suffices to start from the nth power of (32). For brevity, we leave their properties and analytic expressions for a future paper [25].

7. Final Comments and Summary

In this Letter, a nonlinear model of the electromagnetic field is presented, which is based on the fact that Maxwell equations can be derived from an underlying topological structure. Its main characteristics are:

(i) The fundamental field is a complex scalar ϕ defining a map $R^3 \cup \{\infty\} \to C$ at any time or, equivalently via stereographic projection, a map between the spheres $S^3 \to S^2$. Consequently, the solutions are classified in homotopy classes labeled by their Hopf index.

(ii) Although the basic Euler–Lagrange equations are highly nonlinear, the electric and magnetic fields, which are deduced from ϕ by means of (1) and (4), verify Maxwell equations. In other words, (1) is a transformation which linearizes the theory. Not being invertible, however, not all Maxwell solutions can be accepted. This implies a weak form of C-integrability [11].

(iii) If the vectors **E** and **B** are small, this model is equivalent to standard Maxwell theory plus the condition $\mathbf{E} \cdot \mathbf{B} = 0$, because the noninvertibility of the transformation can be neglected in that case, so that all weak field standard electromagnetic fields are admissible.

(iv) The topological charge given by the Hopf index has the same expression as what is called in plasma physics 'the magnetic helicity' [20, 21] (up to a constant action factor) and is quite akin to the helicity used in fluid dynamics [22].

(v) Because of this underlying structure, Maxwell equations in vacuum have topological solutions in which any pair of magnetic lines is linked n times, n being a constant of the motion. If $n = 1$, it seems proper to call them 'quasiphotons'. The individuality of these Hopf knots is maintained by the linking or knottedness of their field strength lines.

The model seems interesting and worthy of further study. It is also noteworthy that some of the strong field solutions of Maxwell are not acceptable or, in other words, that the phase space is smaller.

Although admittedly this is a somewhat imprecise comment, it is worth mentioning here the many problems which quantum electrodynamics must face because of the divergence of integrals at high energy, so that a reduction of phase space might perhaps be welcome, since it could make some integrals converge.

An observation on the appearance of the action constant a is in order. Because of the topological structure, there are elementary excitations which cannot be deformed into the trivial solutions following a continuous path (they are not homotopic to $F_{\mu\nu} = 0$), the action constant a indicating the quantitative importance of these excitations. In this way, the topology induces a quantum-like structure, since, according to (20), all the solutions verify

$$\int \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}^3\mathbf{r} = na, \tag{33}$$

where n is the Hopf index. The magnetic helicity is thus an integer multiple of the action constant, in spite of the fact that the theory is classical in the sense that it makes use of only c-number fields and no second quantization has been performed.

All this will be studied in a future paper [25].

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