Tying Knots in Light Fields

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We construct analytically, a new family of null solutions to Maxwell's equations in free space whose field lines encode all torus knots and links. The evolution of these null fields, analogous to a compressible flow along the Poynting vector that is shear free, preserves the topology of the knots and links. Our approach combines the construction of null fields with complex polynomials on $\mathbb{S}^3$. We examine and illustrate the geometry and evolution of the solutions, making manifest the structure of nested knotted tori filled by the field lines.

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Knots and the application of mathematical knot theory to space-filling fields are enriching our understanding of a variety of physical phenomena with examples in fluid dynamics [1–3], statistical mechanics [4], and quantum field theory [5], to cite a few. Knotted structures embedded in physical fields, previously only imagined in theoretical proposals such as Lord Kelvin's vortex atom hypothesis [6], have in recent years become experimentally accessible in a variety of physical systems, for example, in the vortex lines of a fluid [7–9], the topological defect lines in liquid crystals [10,11], singular lines of optical fields [12], magnetic field lines in electromagnetic fields [13–15], and in spinor Bose-Einstein condensates [16]. Furthermore, numerical simulations have shown that stable knotlike structures arise in the Skyrme-Faddeev model [17,18], and consequently in triplet superconductors [19,20] and charged Bose condensates [21]. Analytical solutions for such excitations, however, are difficult to construct owing to the inherent nonlinearity in most dynamical fields and have therefore remained elusive.

An exception is a particularly elegant solution to Maxwell's equations in free space (see Fig. 1), brought to light by Rañada [22], which provides an encouraging manifestation of a persistent nontrivial topological structure in a linear field theory. This solution, referred to as the Hopfion solution for the rest of the Letter, can furthermore be experimentally realized using tightly focused Laguerre-Gaussian beams [14].

In this Letter, we present the first example of a family of exact knotted solutions to Maxwell's equations in free space, with the electric and magnetic field lines encoding all torus knots and links, which persist for all time. The unique combination of experimental potential and opportunity for analytical study makes light an ideal candidate for studying knotted field configurations and furthermore, a means of potentially transferring knottedness to matter.

In the case of the Hopfion solution illustrated in Fig. 1, the electric, magnetic, and Poynting field lines exhibit a remarkable structure known as a Hopf fibration, with each field line forming a closed loop such that any two loops are linked. At time $t = 0$, each of the electric, magnetic, and Poynting field lines have identical structure (that of a Hopf fibration), oriented in space so that they are mutually orthogonal to each other. The topology of these structures is preserved with time, as the electric and magnetic field lines evolve like unbreakable filaments embedded in a fluid flow, stretching and deforming while retaining their identity [15,23]. The Poynting field lines evolve instead via a rigid translation along the z axis. The Hopfion solution has been rediscovered and studied in several contexts [14,22,24–27] and can be constructed in many ways using complex scalar maps, spinors, twistors.

Despite numerous attempts at generalizing the Hopfion solution to light fields encoding more complex knots, the problem of constructing light fields encoding knots that are preserved in time has remained open until now. Attempts at generalizing Hopfions to torus knots [14,15,28] succeeded at constructing such solutions at an instant in time, but their structure was not preserved [15], and unraveled with time. Beyond Maxwell's equations, the more general problem of finding explicit solutions to dynamical flows which embody persistent knots has also remained open.

The fluidlike topology-preserving evolution of the Hopfion solution is closely tied to the property that the electric and magnetic fields are everywhere perpendicular and of equal magnitude [a constraint known as nullness, cf. Eq. (3)]. Nullness introduces an effective nonlinearity in the problem and imposes a dynamical geometric constraint on Maxwell fields, restricting the space of possible topological configurations of field lines.

We construct knotted solutions within the space of null field configurations by making use of formalisms developed for the construction of null Maxwell fields, such as Bateman's method [29] or equivalently a spinor formalism (see Supplemental Material [30]). The combination of a null electromagnetic field formalism with a topological

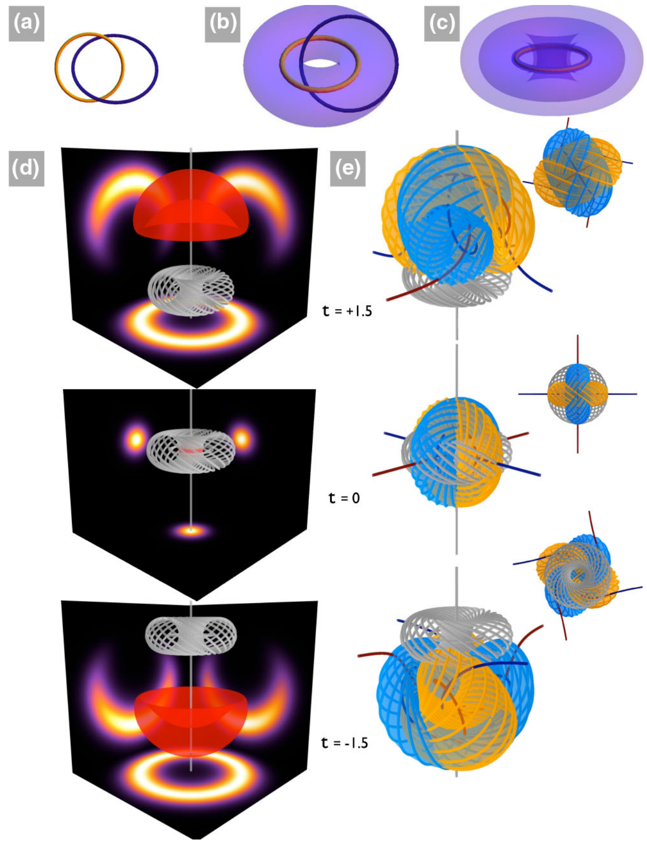

FIG. 1 (color online). Hopfion solution: field line structure (a)–(c) and time evolution (d)–(e). Field lines fill nested tori, forming closed loops linked with every other loop. (a) Hopf link formed by the circle at the core (orange) of the nested tori, and one of the field lines (blue). (b) The torus (purple) that the field line forming the Hopf link is tangent to. (c) Nested tori (purple) enclosing the core, on which the field lines lie. (d) Time evolution of the Poynting field lines (gray), an energy isosurface (red), and the energy density (shown via projections). (e) Time evolution of the electric (yellow), magnetic (blue), and Poynting field lines (gray), with the top view shown in the inset.

construction, leading to a family of knotted null solutions is the central result of this Letter. We now briefly review the key features of the evolution of null electromagnetic fields.

Null electromagnetic fields.—Null electromagnetic fields have a rich history, from the early construction by Bateman [29] to Robinson's theorem [31] and Penrose's twistor theory [32]. For a null electromagnetic field, the Poynting field not only guides the flow of energy, but also governs the evolution of the electric and magnetic field lines. These field lines evolve as though embedded in a fluid, flowing at the speed of light, in the direction of the Poynting field [15,23]. The persistence of the null conditions guarantees the continued fluidlike evolution of the electric and magnetic field lines, giving them the appearance of unbreakable elastic filaments.

The preservation of the null conditions requires the continued perpendicularity of the electric and magnetic fields as they evolve, thus requiring the flow transporting the field lines to be free of shear. Robinson's theorem [31] guarantees the existence of a shear-free family of light rays associated with every null electromagnetic field. In flat space-time, this shear-free family of light rays is given by the normalized Poynting field: the velocity field of the flow transporting the field lines.

The Hopfion solution illustrated in Fig. 1 beautifully demonstrates the features of a null electromagnetic field: the electric and magnetic field lines evolve smoothly, preserving the topology of the field line structure (a Hopf fibration in both cases). The shear-free family of light rays associated with the Hopfion solution, remarkably, also has the structure of a Hopf fibration, which remains unchanged as it evolves with time. This family of light rays is well known in the literature as the Robinson congruence [32].

Since the null condition makes the design of a knotted magnetic or electric field, a problem of engineering a triplet of mutually orthogonal fields that remains orthogonal under time evolution, we start with the formalisms developed for the construction of null fields and seek to construct knotted structures within them. We now briefly summarize Bateman's method for constructing null electromagnetic fields.

Bateman's construction.—Bateman [29] constructs all null electromagnetic fields associated with the same underlying normalized Poynting field, using two complex scalar functions of space-time. Hogan [33], has shown that all null electromagnetic fields can be constructed using Bateman's method.

According to Bateman's construction, given a pair of complex scalar functions of space-time (α, β) which satisfy

$$\boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla}\beta = i(\partial_t\alpha\boldsymbol{\nabla}\beta - \partial_t\beta\boldsymbol{\nabla}\alpha), \quad (1)$$

there is a corresponding electromagnetic field

$$\mathbf{F} = \mathbf{E} + i\mathbf{B} = \nabla\alpha \times \nabla\beta, \quad (2)$$

where $\mathbf{F}$ is known as the Riemann-Silberstein vector [34]. This field is null (both invariants vanish),

$$\mathbf{E}\cdot\mathbf{B} = 0, \qquad \mathbf{E}\cdot\mathbf{E} - \mathbf{B}\cdot\mathbf{B} = 0, \quad (3)$$

since the scalar product $\mathbf{F}\cdot\mathbf{F}$ is zero, as can be seen by taking the dot product of the left-hand side of Eq. (1) with its right-hand side. For the null solutions generated by Eq. (2) to be nontrivial, the following conditions must be satisfied: $\partial_t\alpha((\partial_t\alpha)^2 - (\boldsymbol{\nabla}\alpha)^2) = \partial_t\beta((\partial_t\beta)^2 - (\boldsymbol{\nabla}\beta)^2) = 0$

Each pair (α, β) satisfying Eq. (1) generates a whole family of fields because any vector field of the form

$$\mathbf{F} = h(\alpha, \beta)\boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla}\beta = \boldsymbol{\nabla}f(\alpha, \beta) \times \boldsymbol{\nabla}g(\alpha, \beta), \quad (4)$$

where $h := \partial_\alpha f \partial_\beta g - \partial_\beta f \partial_\alpha g$ and f, g are arbitrary holomorphic functions of (α, β), is a null electromagnetic field. Note that all fields constructed in this way have, by construction, the same normalized Poynting field: $\mathbf{E}\times\mathbf{B}/|\mathbf{E}\times\mathbf{B}| = i(\mathbf{F}\times\mathbf{F}^*/\mathbf{F}\cdot\mathbf{F}^*)$, where $\mathbf{F}^*$ is the complex conjugate of $\mathbf{F}$. This is made manifest, when these null fields are expressed in the equivalent language of spinors (see Supplemental Material [30]).

We list here two simple examples of this construction [27]: a circularly polarized plane wave traveling in the $+z$ direction and the Hopfion solution. They arise from the following choices of α and β. For the plane wave: $\alpha = z - t$, $\beta = x + iy$, $f = e^{i\alpha}$, $g = \beta$, giving $\mathbf{F}^{\mathrm{pw}} = (\hat{x} + i\hat{y})e^{i(z-t)}$. For the Hopfion we have instead $\alpha = -d/b$, $\beta = -ia/(2b)$, $f = 1/\alpha^2$, $g = \beta$ giving $\mathbf{F}^{\mathrm{hp}} = d^{-3}(b^2 - a^2, -i(a^2 + b^2), 2ab)$, where $a = x - iy$, $b = t - i - z$, $d = r^2 - (t - i)^2$.

We now present a family of light-beam-like propagating solutions to Maxwell's equations in free space, in which the electric and magnetic fields encode torus knots and links that are preserved in time. We construct these solutions using complex scalar maps in the context of Bateman's framework. We then describe the knotted structure of the field lines, and compute the entire set of conserved currents, the helicity and charges for electromagnetism in free space, for this family of solutions.

Constructing knotted null electromagnetic fields.— There is a natural connection between knots and singular points of complex maps from $\mathbb{S}^3$ to $\mathbb{C}$. This was used, for example, in recent work by Dennis *et al.* [12] to construct knotted optical vortices in light beams. In particular, given a pair of complex numbers (u, v) such that $|u|^2 + |v|^2 = 1$ (and hence they define coordinates on $\mathbb{S}^3$), it has been shown [35,36] that $u^p \pm v^q = 0$ is the equation of a (p, q) torus knot, when (p, q) are coprime integers.

We note that the following choice of (α, β) in Bateman's construction:

$$\alpha = \frac{r^2 - t^2 - 1 + 2iz}{r^2 - (t - i)^2}, \qquad \beta = \frac{2(x - iy)}{r^2 - (t - i)^2}, \tag{5}$$

which satisfies Eq. (1), admits a natural interpretation as coordinates on $\mathbb{S}^3$ since $|\alpha|^2 + |\beta|^2 = 1$ for any t. At $t = 0$, $(\alpha, \beta) = (u, v)$, the standard stereographic coordinates on $\mathbb{S}^3$.

Hence by [35,36], $\alpha^p \pm \beta^q = 0$ encodes a singular line tied into a (p, q) torus knot when p, q are coprime integers. Guided by this result, we make the intuitive choice of $f(\alpha, \beta) = \alpha^p$ and $g(\alpha, \beta) = \beta^q$ in (4), to obtain the following family of knotted null solutions:

$$\mathbf{F} = \nabla\alpha^p \times \nabla\beta^q, \tag{6}$$

which can equivalently be expressed in terms of spinors (see Supplemental Material [30]). On inspection, we find that the electric and magnetic field lines (shown in Fig. 2) are grouped into knotted and linked tori, nested one inside the other, with (p, q)-torus knots at the core of the foliation. Being smooth, finite energy solutions to Maxwell's equations, these knotted electromagnetic fields are physically feasible, and nullness guarantees that the topology of these knotted structures is preserved in time (as shown in Fig. 3). The shear-free family of light rays associated with this family of solutions is the Robinson congruence.

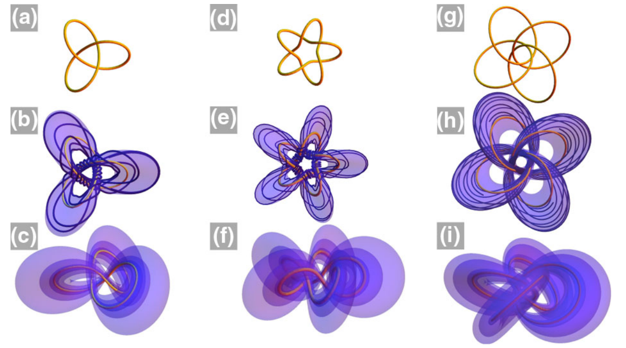

FIG. 2 (color online). Structure of magnetic field lines, (a)–(c) Trefoil knots ($p = 2$, $q = 3$), (d)–(f) Cinquefoil knots ($p = 2$, $q = 5$), (g)–(i) 4 linked rings ($p = 2$, $q = 2$). (a),(d),(g) Core (orange) field line(s) forming (a) a trefoil knot (d) a cinquefoil knot (g) 4 linked rings. (b),(e),(h) Field line(s) (blue) wrapping around the core (orange) confined to a knotted torus (purple) enclosing the core. (c),(f),(i) Knotted nested tori (purple) enclosing the core, on which the field lines lie.

As illustrated in Fig. 2, the magnetic field lines organize around a set of core magnetic field lines, which form (p, q)-torus knots, and stay confined on the surfaces of nested tori, which are isosurfaces of $\Psi_B = \mathrm{Re}\{\alpha^p\beta^q\}$. The innermost core of these nested tori has zero thickness, and corresponds to the knotted core magnetic field lines. Starting from the core, the tori successively increase in thickness (as shown in Fig. 2), until they collide (when $\Psi_B = 0$) and extend to infinity. Since the magnetic field is divergence free and does not vanish on any isosurface $\Psi_B \neq 0$, it follows [37] that all magnetic lines are either periodic or quasiperiodic on each toroidal surface.

As the field evolves, the nested tori along with the knotted core deform smoothly, rotating and stretching, as illustrated in Fig. 3, and the supplementary videos [30].

The electric field lines are also confined on the surface of nested tori (isosurfaces of $\Psi_E = \mathrm{Im}\{\alpha^p\beta^q\}$), organizing around a set of knotted core electric field lines, and have exactly the same structure as the magnetic field lines, rotated in space about the z axis by $\pi/(2q)$. For more detailed descriptions and explicit equations describing the core field lines, see Supplemental Material [30].

To further characterize the physical properties of this family of knotted null fields, we compute the helicity and the full set of conserved quantities [14] corresponding to the known (conformal) symmetries of electromagnetism in free space. The nonvanishing currents and charges normalized by the energy are the following: magnetic helicity $\mathcal{H}_m$ = electric helicity $\mathcal{H}_e = 1/(p + q)$, momentum $\mathbf{P} = \mathbf{J}_{\mathrm{SCT}} = (0, 0, (-p)/(p + q))$, angular momentum $\mathbf{L} = (0, 0, q/(p + q))$, where $\mathbf{J}_{\mathrm{SCT}}$ is the current associated with special conformal transformations; see Supplemental Material [30] for explicit expressions.

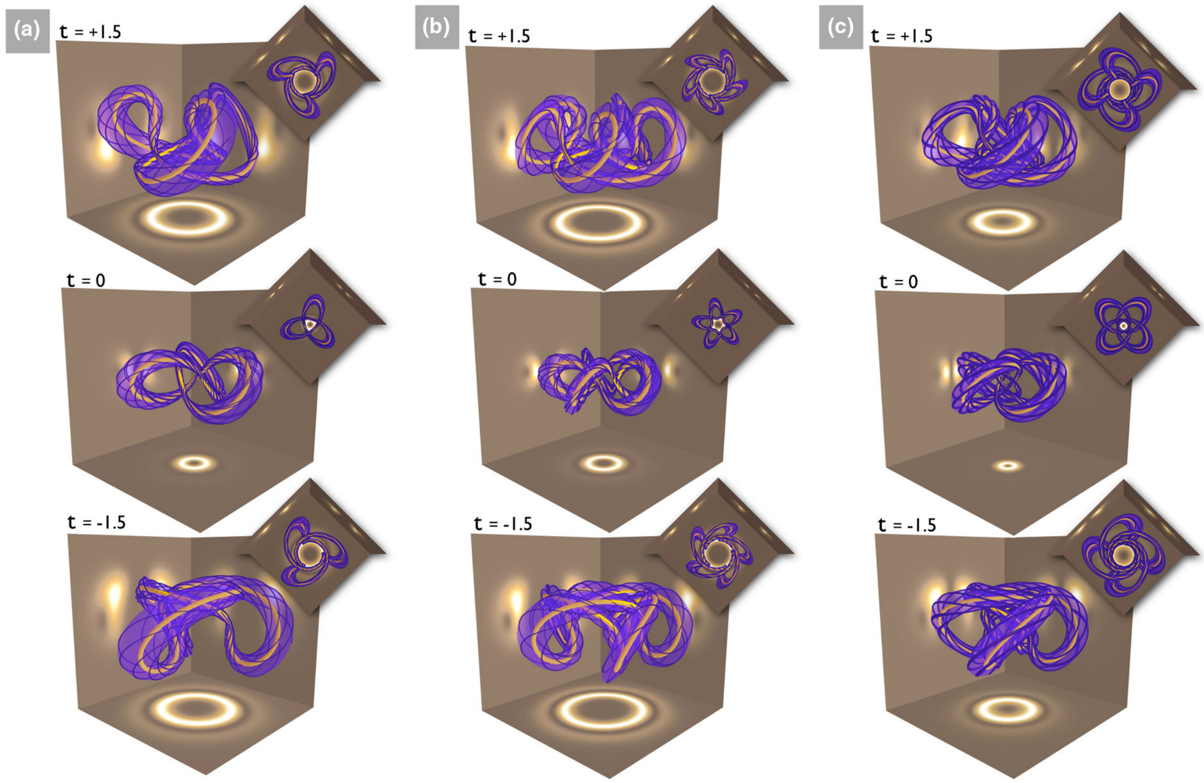

FIG. 3 (color online). Time evolution of magnetic field lines and energy density for (a) the trefoil knot ($p = 2$, $q = 3$), (b) the cinquefoil knot ($p = 2$, $q = 5$), and (c) the 4 Hopf-linked rings ($p = 2$; $q = 2$). Shown is the topology preserving, fluidlike evolution of the core field line(s) (orange), and field line(s) (blue) lying on tori (purple) enclosing the core.

The special choice of $(p, q) = (1, 1)$ yields the Hopfion solution described earlier, in which not only do the core electric and magnetic field lines form Hopf links, but all the other field lines are also closed loops, linked with every other field line.

Summary.—The solutions presented here extend the space of exact, physically feasible, knotted Maxwell fields beyond the Hopfion, by encoding an entire family of both knots and links that are preserved under time evolution. Many open questions remain on the space of knotted states, such as whether solutions with each and every field line knotted and preserved by time evolution exist with topology different from the Hopf fibration (e.g., a Seifert foliation of $\mathbb{S}^3$). Beyond electromagnetism, it remains an open question whether similar explicit solutions can be found for nonlinear evolutions such as the Euler flow of ideal fluids. From a dynamical systems perspective, it may be interesting to explore the role of the invariant tori in the solutions we present and the conditions for which Bateman's construction give rise to electric and magnetic fields with a first integral. Finally, if realized in experiment, can these structures be imprinted on matter such as plasmas or quantum fluids?

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[1] H. K. Moffatt, J. Fluid Mech. **35**, 117 (1969).
[2] H. K. Moffatt, J. Fluid Mech. **106**, 27 (1981).
[3] R. Ricca and M. Berger, Phys. Today **49**, No. 12, 28 (1996).
[4] F. Y. Wu, Rev. Mod. Phys. **64**, 1099 (1992).
[5] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[6] W. Thomson, Philos. Mag. **34**, 15 (1867).
[7] C. F. Barenghi, Milan J. Math. **75**, 177 (2007).
[8] A. Enciso and D. Peralta-Salas, Ann. Math. **175**, 345 (2012).
[9] D. Kleckner and W. Irvine, Nat. Phys. **9**, 253 (2013).
[10] I. I. Smalyukh, Y. Lansac, N. A. Clark, and R. P. Trivedi, Nat. Mater. **9**, 139 (2010).

[11] U. Tkalec, M. Ravnik, S. Copar, S. Zumer, and I. Musevic, Science **333**, 62 (2011).
[12] M. R. Dennis, R. P. King, B. Jack, K. O'Holleran, and M. J. Padgett, Nat. Phys. **6**, 118 (2010).
[13] A. F. Rañada, J. Phys. A **23**, L815 (1990).
[14] W. T. M. Irvine and D. Bouwmeester, Nat. Phys. **4**, 716 (2008).
[15] W. T. M. Irvine, J. Phys. A **43**, 385203 (2010).
[16] Y. Kawaguchi, M. Nitta, and M. Ueda, Phys. Rev. Lett. **100**, 180403 (2008).
[17] L. Faddeev and A. Niemi, Nature (London) **387**, 58 (1997).
[18] P. Sutcliffe, Proc. R. Soc. A **463**, 3001 (2007).
[19] E. Babaev, Phys. Rev. Lett. **88**, 177002 (2002).
[20] E. Babaev, Phys. Rev. B **79**, 104506 (2009).
[21] E. Babaev, L. D. Faddeev, and A. J. Niemi, Phys. Rev. B **65**, 100512 (2002).
[22] A. F. Rañada, Lett. Math. Phys. **18**, 97 (1989).
[23] H. Kedia, D. Peralta-Salas, and W. Irvine (to be published).
[24] A. Trautman, Int. J. Theor. Phys. **16**, 561 (1977).
[25] A. F. Rañada and J. L. Trueba, Phys. Lett. A **202**, 337 (1995).
[26] H. Urbantke, J. Geom. Phys. **46**, 125 (2003).
[27] I. Bialynicki-Birula, J. Opt. A **6**, S181 (2004).
[28] M. Arrayás and J. L. Trueba, arXiv:1106.1122v2.
[29] H. Bateman, *The Mathematical Analysis of Electrical and Optical Wave-Motion* (Dover, New York, 1915).
[30] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.111.150404 for more details on null electromagnetic fields and the family of knotted null Maxwell fields presented here, their equivalent representations in the spinor formalism and videos showing time evolution of the knotted fields.
[31] I. Robinson, J. Math. Phys. (N.Y.) **2**, 290 (1961).
[32] R. Penrose, J. Math. Phys. (N.Y.) **8**, 345 (1967).
[33] P. Hogan, Proc. R. Soc. A **396**, 199 (1984).
[34] I. Bialynicki-Birula, Prog. Opt. **36**, 245 (1996).
[35] K. Brauner, Abh. Math. Semin. Univ. Hambg. **6**, 1 (1928).
[36] J. Milnor, *Singular Points of Complex Hypersurfaces* (Princeton University Press, Princeton, New Jersey, 1969).
[37] S. Sternberg, Am. J. Math. **79**, 397 (1957).

Tying knots in light fields: Supplementary Material

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Brief summary – We describe in more detail, the space of null Maxwell fields and their correspondence with null geodesic shear-free congruences (null GSF congruences), expressing them in the equivalent language of spinors. We then give spinor equivalents for all the fields constructed in the main paper using Bateman's method, showing that the common underlying null GSF congruence for our new family of knotted Maxwell fields is the Robinson congruence. Lastly, we give explicit expressions for all surfaces and lines appearing in Fig. 2–3 in the paper: the knotted core magnetic and electric field lines, and the nested, knotted tori enclosing the core field lines which the rest of the field lines are tangent to, and define the non-vanishing conserved currents and charges calculated for our new family of knotted null fields.

Null electromagnetic fields – A null GSF congruence is a family of null geodesics, described by a null vector field ξ^μ, which satisfies the geodesic and the shear-free conditions [1]. In flat space-time, these conditions are given by:

$$\text{Affine geodesic condition:} \quad \xi^\mu \partial_\mu \xi^\nu = 0 \qquad (1)$$

$$\text{Shear-free condition:} \quad \frac{1}{2}\partial_{(\nu}\,\xi_{\mu)}\partial^\nu \xi^\mu - \left(\frac{1}{2}\partial_\mu\,\xi^\mu\right)^2 = 0 \qquad (2)$$

where, $\partial_{(\nu}\,\xi_{\mu)} = \frac{1}{2}(\partial_\nu \xi_\mu + \partial_\mu \xi_\nu)$.

By Robinson's theorem [2], there is a shear-free family of light rays (a null GSF congruence) underlying each null electromagnetic field. In flat space-time, the null GSF congruence ξ^μ corresponding to a null electromagnetic field is explicitly given by $(1, \frac{\mathbf{E}\times\mathbf{B}}{|\mathbf{E}\times\mathbf{B}|})$.

The correspondence between a null GSF congruence and a null electromagnetic field is not one-to-one, instead a null GSF congruence corresponds to a family of null electromagnetic fields. All null electromagnetic fields can be grouped into such families of null fields, with all null fields in a family corresponding to a common underlying null GSF congruence. In Bateman's formalism, a particular choice of (α, β) determines a null GSF congruence, and the corresponding family of null fields is given by $\mathbf{F} = h(\alpha,\beta)\,\nabla\alpha \times \nabla\beta$, where h is an arbitrary holomorphic function.

This is made manifest when the above null field and its associated null GSF congruence are expressed in the equivalent language of spinors [3]. In this formalism, a null congruence ξ^μ is constructed from a spinor field ξ_A, and a null electromagnetic field $F^{\mu\nu}$ is constructed from a symmetric spinor Φ_{AB} as:

$$\xi^\mu = g^{\mu AA'}\ \xi_A\ \bar{\xi}_{A'}\ ;\ F^{\mu\nu} = g^{\mu AA'} g^{\nu BB'}\left(\Phi_{AB}\,\epsilon_{A'B'} + \epsilon_{AB}\,\bar{\Phi}_{A'B'}\right)$$

where $\{\bar{\Phi}_{A'B'}, \bar{\xi}_{A'}\}$ denote the complex conjugates of $\{\Phi_{AB}, \xi_A\}$, and $\epsilon_{AB} = \epsilon_{A'B'}$ is the 2×2 symplectic matrix, $g^{\mu AA'} = (\mathbb{I}, -\sigma_x, \sigma_y, -\sigma_z)/\sqrt{2}$ are the Infeld-van der Waerden symbols [3, 4], σ_i being the Pauli matrices.

A number of expressions simplify in this language: Maxwell's equations become: $g^{\mu AA'}\partial_\mu \Phi_{AB} = 0$ and the null condition is: $\Phi_{AB}\Phi^{AB} = 0$. Finally, the geodesic and the shear-free conditions for ξ^μ can be combined, and simplify to: $\xi^A \xi_B g^{\mu BB'}\partial_\mu \xi_A = 0$.

A null GSF congruence ξ_A then gives rise to a null electromagnetic field $\Phi_{AB} = \kappa\,\xi_A\,\xi_B$ where the complex scalar κ is chosen to satisfy Maxwell's equations. It is easily verified that Φ_{AB} satisfies the null condition.

The Bateman field $\mathbf{F} = \nabla\alpha \times \nabla\beta$ corresponds to $\Phi_{AB} = \kappa\,\xi_A\,\xi_B$ with:

$$\kappa = \frac{\mathrm{i}}{\partial_{\bar{w}}\bar{\alpha}\,\partial_z\bar{\beta} - \partial_z\bar{\alpha}\,\partial_{\bar{w}}\bar{\beta}}\ ,\ \xi_A = \begin{pmatrix} \partial_w\bar{\alpha}\,\partial_{\bar{w}}\bar{\beta} - \partial_{\bar{w}}\bar{\alpha}\,\partial_w\bar{\beta} \\ \partial_{\bar{w}}\,\bar{\alpha}\,\partial_z\bar{\beta} - \partial_z\,\bar{\alpha}\,\partial_{\bar{w}}\bar{\beta} \end{pmatrix} \qquad (3)$$

if $\partial_{\bar{w}}\bar{\alpha}\,\partial_z\bar{\beta} - \partial_z\bar{\alpha}\,\partial_{\bar{w}}\bar{\beta} \neq 0$, otherwise

$$\kappa = \frac{\mathrm{i}}{\partial_w\,\bar{\alpha}\,\partial_z\bar{\beta} - \partial_z\,\bar{\alpha}\,\partial_w\bar{\beta}}\ ,\ \xi_A = \left(\partial_w\,\bar{\alpha}\,\partial_z\bar{\beta} - \partial_z\,\bar{\alpha}\,\partial_w\bar{\beta},\ 0\right),$$

where $w = x + \mathrm{i}y$, and $\{\bar{\alpha}, \bar{\beta}, \bar{w}\}$ denote the complex conjugates of $\{\alpha, \beta, w\}$.

The entire family of null fields given by $\mathbf{F} = h(\alpha,\beta)\,\nabla\alpha \times \nabla\beta$, thus corresponds to ξ_A as given above with κ rescaled by $\bar{h}$, the complex conjugate of h.

We now give the spinor equivalents of the examples given earlier: a circularly polarized plane wave traveling in the $+z$-direction and the Hopfion solution: $\xi_A^{\mathrm{pw}} = (0, -1)$, $\kappa^{\mathrm{pw}} = -\mathrm{e}^{-\mathrm{i}(z-t)}$ and

$$\xi_A^{\mathrm{hp}} = (-\bar{b}, \bar{a})\ ,\ \kappa^{\mathrm{hp}} = \bar{d}^{-3} \qquad (4)$$

where ξ_A, κ differ from those computed using Eq. (3) (with κ rescaled by $\bar{h}$) by factors that leave the product $\Phi_{AB} = \kappa\,\xi_A\,\xi_B$ unchanged; $\{\bar{a}, \bar{b}, \bar{d}\}$ are complex conjugates of $\{a, b, d\}$. The null GSF congruence underlying the Hopfion solution: ξ_A^{hp}, is referred to in the literature as the Robinson congruence.

Torus knots in null electromagnetic fields – In the spinor formalism the fields described by $\mathbf{F} = \nabla\alpha^p \times \nabla\beta^q$ arise from the same spinor as in (4) but different scaling factors κ:

$$\xi_A = (-\bar{b}, \bar{a}),\ \kappa = 4pq\,\bar{\alpha}^{p-1}\,\bar{\beta}^{q-1}\bar{d}^{-3} \qquad (5)$$

where ξ_A, κ have again been simplified leaving the product $\Phi_{AB} = \kappa\xi_A\xi_B$ unchanged. Thus, the entire family of knotted

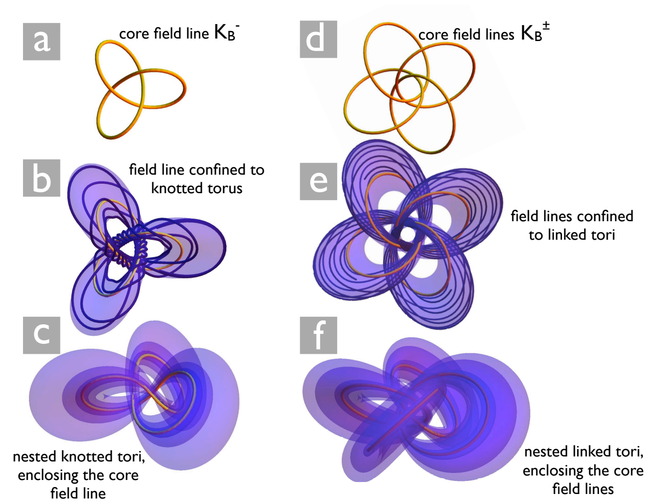

FIG. 1. **Structure of magnetic field lines**, **a-c**: Trefoil knots ($p = 2, q = 3$), **d-f**: 4-Hopf linked rings ($p = 2, q = 2$). **a**: K_B^-, core field line (orange) forming a trefoil knot. **d** Core field lines $K_B^\pm$, each forming a pair of linked rings. **b**,**e**: Field line(s) (blue) wrapping around the core (orange) lying on isosurfaces of Ψ_B which are knotted, linked tori (purple) enclosing the core. **c**,**f**,**i**: Knotted, linked nested tori (purple) enclosing the core given by isosurfaces of Ψ_B.

solutions is constructed by simply changing the scaling factor κ, with the Robinson congruence as the underlying null GSF congruence.

Knotted structure of the field lines – The electric and magnetic field lines organize around corresponding core electric and magnetic field lines, lying on nested tori enclosing these core field lines. Since the electric and magnetic field lines have identical structure, we begin by describing the structure of magnetic field lines, and then go on to describe the structure of electric field lines.

The magnetic field lines are tangent to surfaces of nested tori, which are isosurfaces of $\Psi_B = \mathrm{Re}\{\alpha^p\beta^q\}$, so that: $\mathbf{B}\cdot\nabla\Psi_B = 0$. As Ψ_B is varied from its maximum and minimum values $\left(\pm\sqrt{p^p q^q/(p+q)^{p+q}}\right)$ to zero, the corresponding isosurfaces increase in size, starting from the knotted core magnetic field lines to successively bigger tori, each enveloping all previous ones, until for $\Psi_B = 0$ these tori collide, and the isosurface extends to infinity. The *core* magnetic field lines occupy the loci $K_B^\pm$ of maxima and minima of Ψ_B:

$$K_B^\pm : (\alpha,\beta) = \frac{1}{\sqrt{p+q}}\left(\mathrm{e}^{\mathrm{i}(q\theta+2\pi k/g)}\,\sqrt{p}, \mathrm{e}^{\mathrm{i}(-p\theta-\pi/2q+2\pi k/g\pm\pi/2q)}\,\sqrt{q}\right)$$

where $g = \gcd(p,q)$, $k \in \{0,1,\ldots,g-1\}$ and $\theta \in [0, 2\pi/g)$ parametrizes the curve(s). The curves $K_B^\pm$ lie on a torus, winding p times in the toroidal direction and q times in the poloidal direction, corresponding to β and α changing phase by $-2\pi p$ and $2\pi q$ respectively, thus forming a (p,q)-torus knot for coprime p, q.

The electric field lines have exactly the same structure, rotated in space about the z-axis by $\pi/(2q)$. The corresponding knotted tori that the electric field lines lie on, are isosurfaces of $\Psi_E = \mathrm{Im}\{\alpha^p\beta^q\}$ and the core electric field lines are similarly given by $K_E^\pm : \Psi_E = \pm\sqrt{p^p q^q/(p+q)^{p+q}}$, explicitly given as follows.

$$K_E^\pm : (\alpha,\beta) = \frac{1}{\sqrt{p+q}}\left(\mathrm{e}^{\mathrm{i}(q\theta+2\pi k/g)}\,\sqrt{p}, \mathrm{e}^{\mathrm{i}(-p\theta+2\pi k/g\pm\pi/2q)}\,\sqrt{q}\right)$$

The equations for $K_E^\pm, K_B^\pm$, make it manifest that the knotted core field lines of the electric field $K_E^\pm$ are identical to those of the magnetic field $K_B^\pm$, rotated counterclockwise about the z-axis by $\frac{\pi}{2q}$. We now describe the geometry of the core field lines for all values of the positive integers (p,q):

i. When $(p \neq 1, q \neq 1)$ are coprime, the core field lines are a pair of linked (p,q)-torus knots. One of the core field lines, forming trefoil and cinquefoil knots is shown in Fig. 1 (a).

ii. When $p = 1$ (or $q = 1$), is a pair of linked rings with linking number $2q$ ($2p$), sweeping around the torus q (p) times in the poloidal (toroidal) direction and once in the other direction.

iii. In all other cases, the core field lines comprise of $2g$ linked $(\tilde{p},\tilde{q})$-torus knots if $\tilde{p} \neq 1, \tilde{q} \neq 1$ or $2g$ linked rings otherwise, where $g = \gcd(p,q) \neq 1$, $(p,q) = g * (\tilde{p},\tilde{q})$. For instance, the core magnetic field lines form 4 linked rings for $p = 2, q = 2$ as shown in Fig. 1(d).

This description of the knotted structure of the electric and magnetic field lines holds true for all time, conforming to our expectation of the field line structure of a null field being preserved with time (as seen in Fig. 3 in the manuscript, and the supplementary videos).

Lastly, we list here the expressions for the conserved currents and charges calculated in the paper for our new family of knotted electromagnetic fields. The full set of conserved charges and currents associated with electromagnetism in free space is given in [5] with explicit expressions.

Energy $E := \int \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B}\right)\,\mathrm{d}^3\mathrm{x}$

Magnetic helicity $\mathcal{H}_m := \int \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}^3\mathrm{x}$, where $\nabla\times\mathbf{A} = \mathbf{B}$

Electric helicity $\mathcal{H}_e := \int \mathbf{C}\cdot\mathbf{E}\,\mathrm{d}^3\mathrm{x}$, where $\nabla\times\mathbf{C} = \mathbf{E}$

Momentum $\mathbf{P} := \int \mathbf{E}\times\mathbf{B}\,\mathrm{d}^3\mathrm{x}$

Angular momentum $\mathbf{L} := \int (\mathbf{E}\times\mathbf{B})\times\mathbf{x}\,\mathrm{d}^3\mathrm{x}$

$\mathbf{J}_{SCT} := \int 2\mathbf{x}\,(\mathbf{P}\cdot\mathbf{x}) - 2tE\mathbf{x} - \mathbf{P}\,(\mathbf{x}\cdot\mathbf{x})\,\mathrm{d}^3\mathrm{x}$, where $\mathbf{J}_{SCT}$ is the conserved current associated with the invariance under special conformal transformations (SCT). All integrals in the above expressions are performed over all space.

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[1] H. Bondi, F. A. E. Pirani, and A. Trautman, *Lectures on general relativity, Vol. I* (Brandeis Summer Institute in Theoretical Physics, Prentice-Hall, Englewood Cliffs, NJ, 1965).
[2] I. Robinson, Journal of Mathematical Physics **2**, 290 (1961).

[3] R. Penrose and W. Rindler, *Spinors and Space-Time: Volume 1, Two-Spinor Calculus and Relativistic Fields* (Cambridge University Press, 1987).

[4] L. Infeld and B. L. van der Waerden, S. B. Preuss. Akad. **9**, 380 (1933).

[5] W. T. M. Irvine and D. Bouwmeester, Nature Physics **4**, 716 (2008).