The Golden Section, Fibonacci series, and new hyperbolic models of Nature

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Abstract

Приведен обзор новых математических моделей Природы, основанных на золотом сечении и использовании *гиперболических функций Фибоначчи и Люка* и *функции «Золотой Шофар»*. Также рассматриваются *обобщенные числа Фибоначчи, обобщенные золотые пропорции, обобщенный принцип золотого сечения, «золотые» алгебраические уравнения, обобщенные формулы Бине, непрерывные функции для обобщенных чисел Фибоначчи и Люка, матрицы Фибоначчи и «золотые» матрицы*. Статья опубликована в международном электронном журнале "Visual Mathematics", 2006, Vol. 3, No. 3 http://www.mi.sanu.ac.yu/vismath/stakhov/index.html

A survey of new mathematical models of Nature is presented based on the golden section and using a class of h*yperbolic Fibonacci and Lucas functions,* and a *surface referred to as the Golden Shofar*. Also considered are *generalized Fibonacci numbers, generalized golden proportions, a Generalized Principle of the Golden Section, golden algebraic equations, generalized Binet formulas, generalized Lucas numbers, continuous functions for the generalized Fibonacci and Lucas numbers, Fibonacci matrices,* and *golden matrices*.

Key words. The golden section, Fibonacci and Lucas numbers, Binet formulas, the Golden Shofar, hyperbolic functions, Dichotomy principle, the Golden Section principle.

1. Introduction

One of the major achievements of modern science is an understanding that the world of Nature is hyperbolic. The creator of non-Euclidean geometry was the Russian mathematician Nikolay Lobachevsky who derived a new geometric system based on *hyperbolic functions* in 1827. The need for new geometrical ideas became apparent in physics at the beginning of the 20th century as the result of *Einstein's Special Theory of Relativity* (1905). In 1908, three years after the publication of this great work, the German mathematician Herman Minkowsky gave a geometrical interpretation of the Special Theory of Relativity based on hyperbolic ideas.

The theory of hyperbolic functions has developed in ways that, at first sight, does not appear to have any connection to hyperbolic functions. However, it does relate to the theory of Fibonacci numbers, an actively developing branch of modern mathematics [1-3]. In 1993, the Ukrainian mathematicians Alexey Stakhov and Ivan Tkachenko developed a new approach to the theory of hyperbolic functions [4]. Using the so-called *Binet formulas*, they developed a new class of hyperbolic functions called h*yperbolic Fibonacci and Lucas functions* [4, 5]. This idea was further developed in Stakhov and Rozin's paper [6] where they defined a class of s*ymmetric hyperbolic Fibonacci and Lucas f*unctions*.* In Stakhov and Rozin's article [7] a new surface of the second degree called the *Golden Shofar* was developed. The hyperbolic Fibonacci and Lucas functions and the Golden Shofar surface are the most important ingredients of the *"golden" mathematical models* applicable to the description of the "hyperbolic worlds" of Nature.

Also useful for the study of such models are classes of *generalized Fibonacci numbers* [8], *generalized golden proportions* [8], *golden algebraic equations* [9], *generalized Binet formulas* [10], *generalized Lucas numbers* [10], *continuous functions for the generalized Fibonacci and Lucas functions* [11], *Fibonacci matrices* [12] and *golden matrices* [13]. These mathematical entities were used in *algorithmic measurement theory* [8, 14, 15], *a new theory of real numbers* [16], *harmony mathematics* [17-20], *new computer arithmetics* [8, 21-23], *a new coding theory* [24-25], and the formulation of the *Generalized Principle of the Golden Section* [13].

The main purpose of the present article is to give a brief survey of golden mathematical models used for describing the "hyperbolic worlds" of Nature developed by the authors in references [4-25].

2. The Golden Section, Fibonacci & Lucas numbers, and Binet formulas

In The Elements of Euclid there is a geometric problem to divide *a line segment in the extreme and mean ratio*. This problem is often called the *golden section* problem [1-3] the solution of which reduces to the equation,

$$x^2 = x + 1. \tag{1}$$

Equation (1) has two roots; its positive root $\tau = \dfrac{1+\sqrt{5}}{2}$ is called *golden proportion, golden mean, or golden ratio. The following expression for the golden ratio* follows from Equation (1):

$$\tau^n = \tau^{n-1} + \tau^{n-2} = \tau \times \tau^{n-1} \tag{2}$$

where n is an integer (either positive or negative).

The *Fibonacci numbers*

$$F_n = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\} \tag{3}$$

are an integer sequence given by the recursive relation:

$$F_n = F_{n-1} + F_{n-2} \tag{4}$$

with initial conditions:

$$F_1 = F_2 = 1 \tag{5}$$

The *Lucas numbers*

$$L_n = \{1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots\} \tag{6}$$

are another integer sequence given by the recursive relation:

$$L_n = L_{n-1} + L_{n-2} \tag{7}$$

with initial conditions:

$$L_1 = 1; L_2 = 3 \tag{8}$$

. Fibonacci and Lucas numbers can be extended to negative values of the index n as shown in Table 1

Table 1. The extended Fibonacci and Lucas numbers

N	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	-1	2	-3	5	-8	13	-21	34	-55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	-1	3	-4	7	-11	18	-29	47	-76	123

A number of wonderful mathematical properties of the extended series F_n and L_n follow from Table 1. For example, for odd integers $n = 2k + 1$ the terms of the sequences F_n and F_{-n} coincide, i.e., $F_{2k+1} = F_{-2k-1}$, while for even integers $n = 2k$ they have opposite signs, i.e., $F_{2k} = -F_{-2k}$, whereas the reverse occurs for the Lucas numbers L_n, with $L_{2k} = L_{-2k}$ and $L_{2k+1} = -L_{-2k-1}$.

There are many fundamental results in the modern theory of Fibonacci numbers [1-3]. In the 19th century the famous French mathematician Binet discovered two remarkable formulas, called the *Binet formulas,* that connect Fibonacci (F_n) and Lucas (L_n) numbers with the golden ratio $\tau = \dfrac{1+\sqrt{5}}{2}$.

$$F_n = \begin{cases} \dfrac{\tau^{2k+1} + \tau^{-(2k+1)}}{\sqrt{5}} \\ \dfrac{\tau^{2k} - \tau^{-2k}}{\sqrt{5}} \end{cases}, \qquad (9)$$

$$L_n = \begin{cases} \tau^{2k} + \tau^{-2k} \\ \tau^{2k+1} - \tau^{-(2k+1)} \end{cases}. \qquad (10)$$

for integer values of *k*.

3. Hyperbolic Fibonacci and Lucas functions

3.1. Stakhov and Tkachenko's approach

If we compare Binet formulas (9) and (10) to the classical hyperbolic functions

$$shx = \frac{e^x - e^{-x}}{2}, \qquad (11)$$

$$chx = \frac{e^x + e^{-x}}{2}. \qquad (12)$$

we notice a similarity.

In [4], the discrete variable *k* in formulas (9) and (10) was replaced by the continuous variable *x* that takes its values from the set of real numbers. Consequently, the following continuous functions, which are called *the hyperbolic Fibonacci and Lucas functions,* were introduced:

The hyperbolic Fibonacci sine

$$sF(x) = \frac{\tau^{2x} - \tau^{-2x}}{\sqrt{5}} \qquad (13)$$

The hyperbolic Fibonacci cosine

$$cF(x) = \frac{\tau^{2x+1} + \tau^{-(2x+1)}}{\sqrt{5}} \qquad (14)$$

The hyperbolic Lucas sine

$$sL(x) = \tau^{2x+1} - \tau^{-(2x+1)} \qquad (15)$$

The hyperbolic Lucas cosine

$$cL(x) = \tau^{2x} + \tau^{-2x} \qquad (16)$$

The connection between Fibonacci (F_n) and Lucas (L_n) numbers and the hyperbolic Fibonacci and Lucas functions (13)-(16) is given by the following identities:

$$sF(k) = F_{2k}; \quad cF(k) = F_{2k+1}; \quad sL(k) = L_{2k+1}; \quad cL(k) = L_{2k}, \qquad (17)$$

where *k* is an arbitrary integer.

3.2. A symmetrical representation of the hyperbolic Fibonacci and Lucas functions (Stakhov and Rozin's approach)

Let us compare the hyperbolic Fibonacci and Lucas functions to the classical hyperbolic functions. It is easy to see that, in contrast to the classical hyperbolic functions, the graph of the Fibonacci cosine is asymmetric, while the graph of the Lucas sine is asymmetric relative to the origin of the coordinates. This places restrictions on important applications of the new class of hyperbolic functions given by (13)-(16).

Based on an analogy between Binet formulas (9) and (10) and the classical hyperbolic functions (11) and (12), the following definitions of the hyperbolic Fibonacci and Lucas functions were given by Stakhov and Rozin [6]:

Symmetric hyperbolic Fibonacci sine

$$sFs(x) \;\; = \;\; \frac{\tau^{x} \;\; - \;\; \tau^{-x}}{\sqrt{5}} \tag{18}$$

<p align="center">Symmetric hyperbolic Fibonacci cosine</p>

$$cFs(x) \;\; = \;\; \frac{\tau^{x} \;\; + \;\; \tau^{-x}}{\sqrt{5}} \tag{19}$$

<p align="center">Symmetric hyperbolic Lucas sine</p>

$$sLs(x) \;\; = \;\; \tau^{x} \;\; - \;\; \tau^{-x} \tag{20}$$

<p align="center">Symmetric Lucas cosine</p>

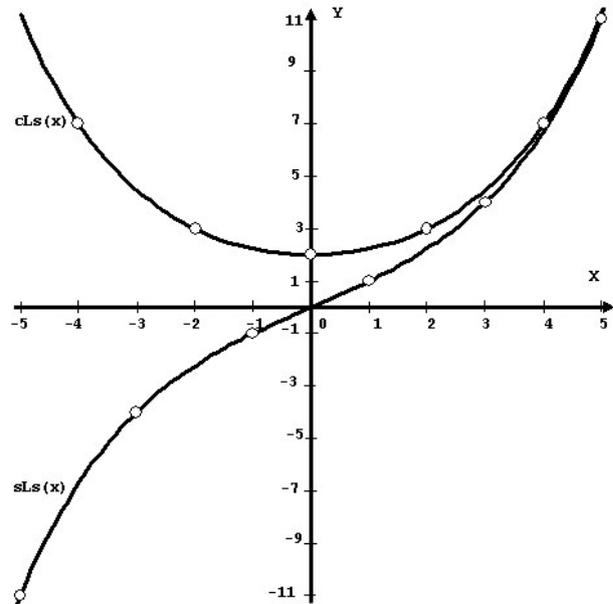$$cLs(x) \;\; = \;\; \tau^{x} \;\; + \;\; \tau^{-x} \tag{21}$$

Fibonacci and Lucas numbers are determined from the symmetric Fibonacci and Lucas functions by the following substitutions,

$$F_n \; = \; \begin{cases} sFs(n), & for \;\; n = 2k \\ cFs(n), & for \;\; n = 2k+1 \end{cases} \; ; \qquad L_n \; = \; \begin{cases} cLs(n), & for \;\; n = 2k \\ sLs(n), & for \;\; n = 2k+1 \end{cases} . \tag{22}$$

The authors refer to the functions defined by (18)-(21) as the *symmetrical representation*.
It is easy to construct graphs of the symmetric Fibonacci and Lucas functions (Fig. 1, 2).
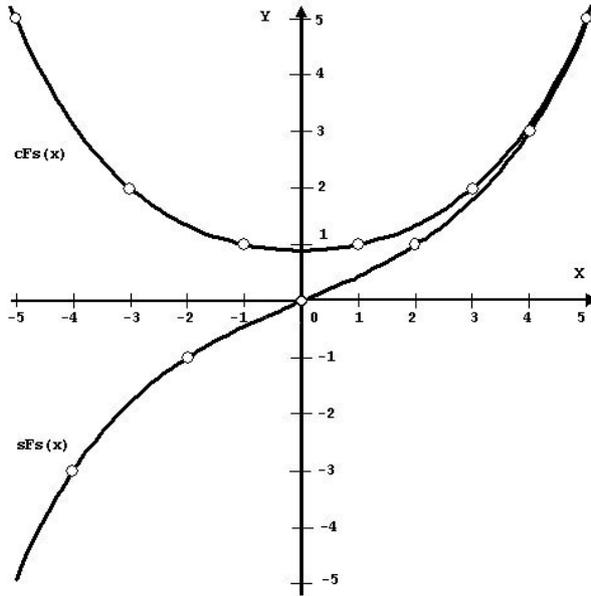
Figure 1. The symmetrical Fibonacci functions	**Figure 2.** The symmetrical Lucas functions

Their graphs have a symmetric form and are similar to the graphs of the classical hyperbolic functions. However, at the point $x=0$, the symmetric Fibonacci cosine $cFs(x)$ takes the value $cFs(0) = \frac{2}{\sqrt{5}}$, while the symmetric Lucas cosine $cLs(x)$ takes the value $cLs(0) = 2$. It is also important to note that the Fibonacci numbers F_n, with even indices, are values of the symmetrical Fibonacci sine, $sFs(x)$, at even integer values of x and the Fibonacci numbers, with odd indices, are values of the symmetric Fibonacci cosine, $cFs(x)$, at odd integers values of x. On the other hand, Lucas numbers, with the even indices, are values of the symmetrical Lucas cosine $cLs(x)$ at the even integers and the Lucas numbers, with the odd indices, are values of the symmetrical Lucas cosine $sLs(x)$ at the odd integers.

The symmetric hyperbolic Fibonacci and Lucas functions are related to the classical hyperbolic functions by the identities:

$$sFs(x) = \frac{2}{\sqrt{5}} sh(\ln(\alpha) \cdot x); \qquad\qquad cFs(x) = \frac{2}{\sqrt{5}} ch(\ln(\alpha) \cdot x);$$

$$sLs(x) = 2 \ sh(\ln(\alpha) \cdot x); \qquad\qquad cLs(x) = 2 \ ch(\ln(\alpha) \cdot x).$$

The symmetric hyperbolic Fibonacci and Lucas functions are related among each other by the identities:

$$sFs(x) = \frac{sLs(x)}{\sqrt{5}} \quad ; \quad cFs(x) = \frac{cLs(x)}{\sqrt{5}} \quad .$$

3.3. The recursive properties of the symmetric hyperbolic Fibonacci and Lucas functions

The symmetric hyperbolic Fibonacci and Lucas functions (18)-(21) are generalizations of Fibonacci and Lucas numbers, and therefore they have *recursive properties*. On the other hand, they are similar to the classical hyperbolic functions, and therefore they have *hyperbolic properties.*

Table 2 lists the well known identities for Fibonacci and Lucas numbers and the corresponding identities for the symmetrical hyperbolic Fibonacci and Lucas functions.

Table 2. The recursive properties of the symmetrical hyperbolic Fibonacci and Lucas functions

The identities for Fibonacci and Lucas numbers	The identities for the symmetric hyperbolic Fibonacci and Lucas functions	
$F_{n+2} = F_{n+1} + F_n$	$sFs(x+2) = cFs(x+1) + sFs(x)$	$cFs(x+2) = sFs(x+1) + cFs(x)$
$F_n{}^2 - F_{n+1} F_{n-1} = (-1)^{n+1}$	$[sFs(x)]^2 - cFs(x+1) \ cFs(x-1) = -1$	$[cFs(x)]^2 - sFs(x+1) \ sFs(x-1) = 1$
$L_{n+2} = L_{n+1} + L_n$	$sLs(x+2) = cLs(x+1) + sLs(x)$	$cLs(x+2) = sLs(x+1) + cLs(x)$
$L_n{}^2 - 2(-1)^n = L_{2n}$	$[sLs(x)]^2 + 2 = cLs(2x)$	$[cLs(x)]^2 - 2 = cLs(2x)$
$F_{n+1} + F_{n-1} = L_n$	$cFs(x+1) + cFs(x-1) = cLs(x)$	$sFs(x+1) + sFs(x-1) = sLs(x)$
$F_n + L_n = 2F_{n+1}$	$cFs(x) + sLs(x) = 2sFs(x+1)$	$sFs(x) + cLs(x) = 2cFs(x+1)$

For example, the famous Cassini formula [26]

$$F_n{}^2 - F_{n+1} F_{n-1} = (-1)^{n+1}, \tag{23}$$

which is an important identity connecting three adjacent Fibonacci numbers, can be generalized to the symmetric hyperbolic Fibonacci functions as,

$$[sFs(x)]^2 - cFs(x+1) \ cFs(x-1) = -1 \tag{24}$$
$$[cFs(x)]^2 - sFs(x+1) \ sFs(x-1) = 1, \tag{25}$$

It is clear that the identities (24), (25) can be considered to be generalizations of the Cassini formula to continuous functions**.**

3.4. The hyperbolic properties of the symmetric hyperbolic Fibonacci and Lucas functions

Some of the hyperbolic properties of the symmetric hyperbolic Fibonacci and Lucas functions have analogues to the properties of classical hyperbolic functions as shown in Table 3.

Table 3. The hyperbolic properties of the symmetric hyperbolic Fibonacci and Lucas functions

Classical hyperbolic function	Symmetric hyperbolic Fibonacci function	Symmetric hyperbolic Lucas function
$[ch(x)]^2 - [sh(x)]^2 = 1$	$[cFs(x)]^2 - [sFs(x)]^2 = \frac{4}{5}$	$[cLs(x)]^2 - [sLs(x)]^2 = 4$
$ch(x\pm y) = ch(x)ch(y) \pm$ $\pm sh(x)sh(y)$	$\frac{2}{\sqrt{5}} cFs(x\pm y) = cFs(x)cFs(y) \pm$ $\pm sFs(x)sFs(y)$	$2cLs(x\pm y) = cLs(x)cLs(y) \pm$ $\pm sLs(x)sLs(y)$
$sh(x\pm y) = sh(x)ch(y) \pm ch(x)sh(y)$	$\frac{2}{\sqrt{5}} sFs(x\pm y) = sFs(x)cFs(y) \pm$ $\pm cFs(x)sFs(y)$	$2sLs(x\pm y) = sLs(x)cLs(y) \pm$ $\pm cLs(x)sLs(y)$

$ch(2x) = [ch(x)]^2 + [sh(x)]^2$	$\dfrac{2}{\sqrt{5}} cFs(2x) = [cFs(x)]^2 + $ $+[sFs(x)]^2$	$2cLs(2x) = [cLs(x)]^2 + $ $+[sLs(x)]^2$
$Sh(2x) = 2\,sh(x)ch(x)$	$\dfrac{1}{\sqrt{5}} sFs(2x) = sFs(x)cFs(x)$	$sLs(2x) = sLs(x)cLs(x)$
$[ch(x)]^{(n)} = \begin{cases} sh(x), & for \quad n = 2k+1 \\ ch(x), & for \quad n = 2k \end{cases}$	$[cFs(x)]^{(n)} = $ $= \begin{cases} (\ln(\tau))^n sFs(x), & for \quad n = 2k+1 \\ (\ln(\tau))^n cFs(x), & for \quad n = 2k \end{cases}$	$[cLs(x)]^{(n)} = $ $= \begin{cases} (\ln(\tau))^n sLs(x), & for \quad n = 2k+1 \\ (\ln(\tau))^n cLs(x), & for \quad n = 2k \end{cases}$

For example, the most important identity for the classical hyperbolic functions,
$$[ch(x)]^2 - [sh(x)]^2 = 1 \tag{26}$$
is written for the hyperbolic Fibonacci and Lucas functions as :
$$[cFs(x)]^2 - [sFs(x)]^2 = \frac{4}{5}, \tag{27}$$
$$[cLs(x)]^2 - [sLs(x)]^2 = 4. \tag{28}$$

Thus, the above symmetric hyperbolic functions retain all of the properties of classical hyperbolic functions (Table 3) while exhibiting new (recursive) properties characteristic of Fibonacci and Lucas numbers (Table 2). Thus, unlike the classical hyperbolic functions, the new hyperbolic functions have discrete analogues in the form of Fibonacci and Lucas numbers. According to (22) these functions agree with Fibonacci and Lucas numbers when the continuous variable x takes integer values. Note that identities (24) - (28), as well as other identities from Tables 2 and 3 emphasize the fundamental character of the hyperbolic Fibonacci and Lucas functions.

We predict that hyperbolic Fibonacci and Lucas functions will have great importance for the future development of Fibonacci and Lucas number theory [1-3]. They generalize Fibonacci and Lucas numbers to the continuous domain since Fibonacci and Lucas numbers are embedded in them. According to Table 2 each discrete identity for Fibonacci and Lucas numbers has its continuous analogue in the form of a corresponding identity for the hyperbolic Fibonacci and Lucas functions, and conversely. Therefore, the theory of the hyperbolic Fibonacci and Lucas functions is more general than traditional Fibonacci and Lucas number theory. **This results in the following important consequence**: **due to the introduction of hyperbolic Fibonacci and Lucas functions, discrete Fibonacci and Lucas number theory [1-3] becomes a part of the continuous theory of hyperbolic Fibonacci and Lucas functions [4-7]! The introduction of hyperbolic Fibonacci and Lucas functions is a new stage in the development of Fibonacci and Lucas number theory.**

3.5. Bodnar's geometry

As is well known, Fibonacci and Lucas numbers are the basis of *Law of Plant Phyllotaxis* [27]. According to this law, the number of the left and right spirals on the surface of *phyllotaxis objects* such as the pine cone, pineapple, cactus, head of sunflower, etc. are adjacent Fibonacci numbers, i.e., numbers whose ratios are,
$$\frac{F_{n+1}}{F_n} : \quad \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, ... \to \tau = \frac{1+\sqrt{5}}{2}. \tag{29}$$

These ratios characterize the symmetry of phyllotaxis. Therefore, every phyllotaxis pattern is characterized by one of these ratios. This ratio is called the *order of symmetry*.

After observing phyllotaxis patterns on the surface of a plant, it is natural to wonder how this pattern is formed during the growth of the plant. This question is at the basis of the *phyllotaxis puzzle* which is one of the most intriguing problems of botany. The key to this puzzle lies in the tendency of bioforms during their growth to change their symmetry orders according to (29). For example, the florets of a sunflower at the different levels of the same stalk have different symmetry orders, that is, the older disks have symmetry orders further out in the sequence. It means that during growth there is a natural change f the symmetry order and this change of symmetry is carried out according to the law:

$$\frac{2}{1} \rightarrow \frac{3}{2} \rightarrow \frac{5}{3} \rightarrow \frac{8}{5} \rightarrow \frac{13}{8} \rightarrow \frac{21}{13} \rightarrow \dots \qquad (30)$$

A change of symmetry orders of phyllotaxis objects according to (30) is called *dynamic symmetry* [27]. Many scientists feel that the phyllotaxis phenomenon has more general importance. For example, it is Vernadsky 's opinion that dynamic symmetry may be applicable to general problems in biology.

So, the phenomenon of the dynamic symmetry finds a special role in the geometry of phyllotaxis. We can assume that certain geometrical laws are lurking within the numerical regularity of (30). This phenomenon may be the secret of the phyllotaxis growth mechanism, and its expression could have great importance for the solution of the phyllotaxis problem. The Ukrainian researcher Oleg Bodnar has recently shed light on this problem [27]. Bodnar formulated a geometric theory of phyllotaxis**.** This theory is based on the hypothesis, that the geometry of phyllotaxis is hyperbolic, and the change of symmetry orders of phyllotaxis objects during their growth is based on the hyperbolic turn, the basic transforming motion of hyperbolic geometry. However, the main feature of Bodnar's geometry lies in its use of golden hyperbolic functions, which agree with the symmetric hyperbolic Fibonacci and Lucas functions described in this paper up to constant factors.

4. The Golden Shofar

4.1. The quasi-sine Fibonacci function

Let us consider the representation of the Binet formula for the Fibonacci numbers in the following form widely used in mathematics [1-3]:

$$F_n \;=\; \frac{\tau^n \;-\; (-1)^n \; \tau^{-n}}{\sqrt{5}} \;, \qquad (31)$$

where *n is an arbitrary integer.*

Comparing the Binet formula to the symmetric hyperbolic Fibonacci functions (19) and (20), it follows that the continuous functions τ^x and τ^{-x} in the formulas (19) and (20) correspond to the discrete sequences τ^n and τ^{-n} in Equation (31). Consequently it is possible to insert a continuous function into this function that takes the values -1 and 1 at integer values of n analogous to the alternating sequence $(-1)^n$ in Binet's formula**.** The trigonometric function $\cos(\pi x)$ is the simplest such function. This suggests a new set of continuous functions that are connected to Fibonacci numbers.

Definition 1. The following continuous function is called the ***quasi-sine Fibonacci function***:

$$FF(x) \;=\; \frac{\tau^x \;-\; \cos(\pi x) \; \tau^{-x}}{\sqrt{5}} \;. \qquad (32)$$

where,

$$F_n = FF(n) \;=\; \frac{\tau^n \;-\; \cos(\pi n) \; \tau^{-n}}{\sqrt{5}} \;, \qquad (33)$$

for integer values of *n*.

The graph of the quasi-sine Fibonacci function passes through all points of the coordinate plane corresponding to the Fibonacci numbers (see Fig.3). The symmetric hyperbolic Fibonacci functions (19) and (20) (Fig. 1) are the envelopes of the quasi-sine Fibonacci function .

Also the quasi-sine Fibonacci function has recursive properties similar to Fibonacci numbers [7].
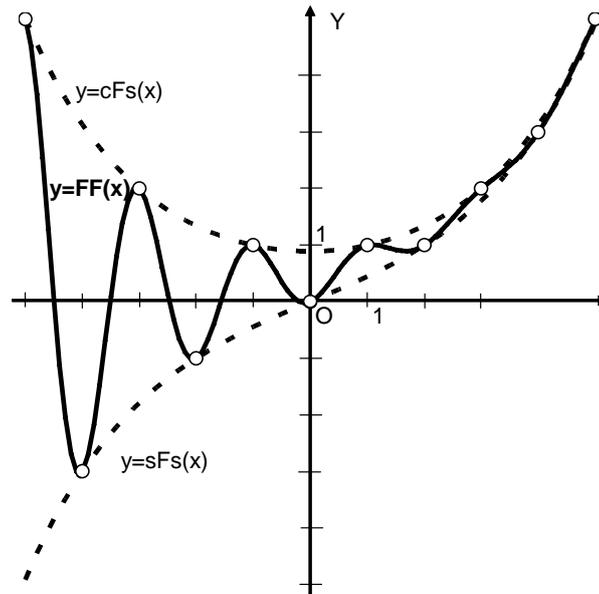
Figure 3. The graph of the quasi-sine Fibonacci function
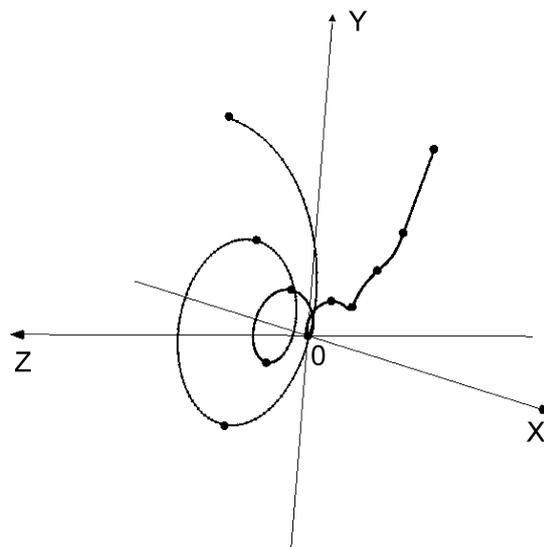
4.2. The three-dimensional Fibonacci spiral

It is well known that the trigonometric sine and cosine can be defined as the projection of the translational movement of a point on the surface of an infinite rotating cylinder with the radius 1, whose symmetry center coincides with the x-axis. Such a three-dimensional spiral is projected into the sine function on a plane and described by the complex function $f(x) = \cos(x) + i\sin(x)$, where $i = \sqrt{-1}$.

If we assume that the quasi-sine Fibonacci function (32) is a projection of a three-dimensional spiral on some funnel-shaped surface, then analogous to the trigonometric sine, it is possible to construct a *three-dimensional Fibonacci spiral*.

Definition 2. The following function is a complex representation of *the three-dimensional Fibonacci spiral*:

$$CFF(x) = \frac{\tau^x - \cos(\pi x) \ \tau^{-x}}{\sqrt{5}} + i\frac{\sin(\pi x) \ \tau^{-x}}{\sqrt{5}}. \tag{34}$$

This function, by its shape, resembles a spiral that is drawn on a crater with a bent end (Fig. 4).

Figure 4. The three-dimensional Fibonacci spiral

Consider real and imaginary parts of the complex Fibonacci spiral:

$$\text{Re}[CFF(x)] = \frac{\tau^x - \cos(\pi x) \ \tau^{-x}}{\sqrt{5}} ; \qquad (35)$$

$$\text{Im}[CFF(x)] = \frac{\sin(\pi x) \ \tau^{-x}}{\sqrt{5}} . \qquad (36)$$

Designating the real part by $y(x)$ and the imaginary part by $z(x)$, we can get the following system of equations:
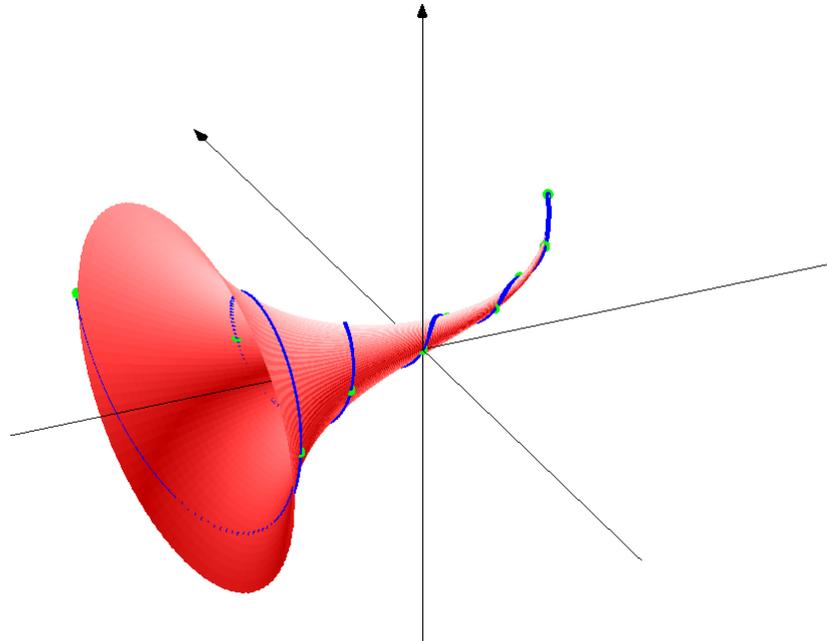
$$\begin{cases} y(x) - \dfrac{\alpha^x}{\sqrt{5}} = - \dfrac{\cos(\pi x) \ \alpha^{-x}}{\sqrt{5}} , \\[3mm] z(x) \qquad\quad = \dfrac{\sin(\pi x) \alpha^{-x}}{\sqrt{5}} . \end{cases} \qquad (37)$$

Square both expressions of system (37) and add them, taking y and z as independent variables, to get the following curvilinear surface.

Definition 3. The three-dimensional curved surface given by,

$$\left(y - \frac{\tau^x}{\sqrt{5}} \right)^2 + z^2 = \left(\frac{\tau^{-x}}{\sqrt{5}} \right)^2 . \qquad (38)$$

is called the **Golden Shofar** and shown in Fig. 5..

Figure 5. The Golden Shofar

The Golden Shofar looks like a horn or crater with its narrow end bent up. Translating from the Hebrew language, the word "Shofar" means a horn that is a symbol of power or might. The Shofar is blown on The Judgment Day (the Jewish New Year) and the Day of Atonement (the Yom Kippur).

Formula (38) for the Golden Shofar can also be represented in the form:

$$z^2 = [cFs(x) - y][sFs(x) + y], \qquad (39)$$

where $sFs(x)$ and $cFs(x)$ are the symmetric hyperbolic Fibonacci functions (20) and (21) respectively.

4. A general model of hyperbolic space with a "shofarable" topology

Based on experimental data obtained in 2003 by the NASA's Wilkinson Microwave Anisotropy Probe (WMAP**),** a new hypothesis about the structure of the Universe was developed. According to [28], the geometry of the Universe is similar in shape to a horn or a pipe with an extended bell [27]. As a result of this disclosure we make the following claim:

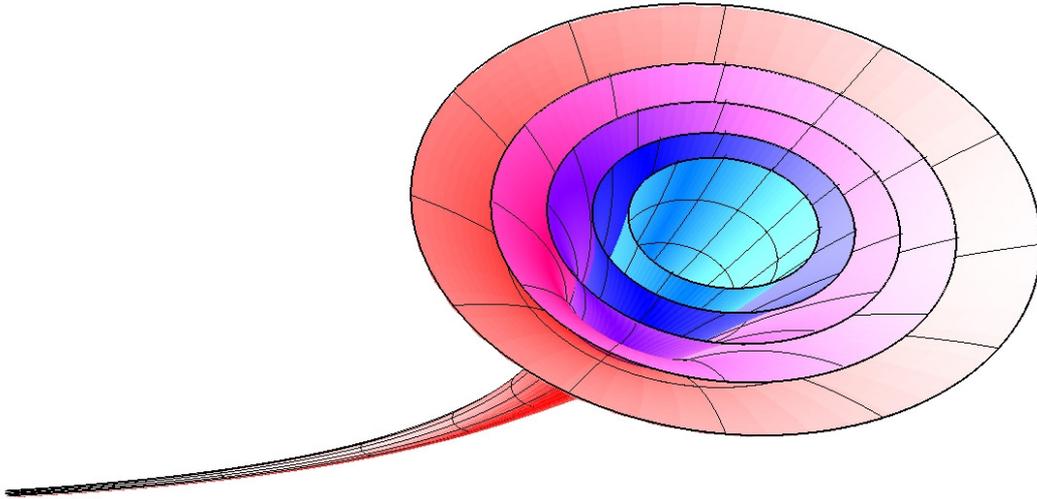The Universe has a "shofarable" topology as shown in Fig.6.

Figure 6. The "shofarable" topology

6. A further development of the golden mathematical models

6.1. Fibonacci p-numbers

Reference [8] introduces *generalized Fibonacci numbers* related to Pascal's triangle. For a given p=0, 1, 2, 3, ... generalized Fibonacci numbers are given by the following recursive relation:

$$F_p(n+1) = F_p(n)+F_p(n\text{-}p) \quad \text{for} \quad n>p+1; \tag{40}$$

$$F_p(1) = F_p(2) = ... = F_p(p+1) = 1 \tag{41}$$

Integer sequences corresponding to formulas (40) and (41) are called *Fibonacci p-numbers[8]*.

We have proved that Fibonacci p-numbers can be expressed in terms of the binomial coefficients as follows [8]:

$$F_p(n+1) = C_n^0 + C_{n-p}^1 + C_{n-2p}^2 + C_{n-3p}^3 + C_{n-4p}^4 + ... \tag{42}$$

Note that expressions (40) and (41) create an infinite number of new recursive numerical sequences since every p generates its own numerical sequence. In particular, p=0 generates the binary sequence: 1, 2, 4, 8, 16, 32, ... , while for p=1 the classical Fibonacci sequence (3) results. For the case p=0 Equation (42) takes the form:

$$C_n^0 + C_n^1 + ... + C_n^n = 2^n \tag{43}$$

widely used in combinatorial analysis. For p=1,

$$F_1(n+1) = C_n^0 + C_{n-1}^1 + C_{n-2}^2 + C_{n-3}^3 + C_{n-4}^4 + ..., \tag{44}$$

This formula relates Fibonacci numbers to the binomial coefficients.

6.2. The golden p-proportions and the golden algebraic equations

It is well known that the ratios of adjacent Fibonacci numbers F_n/F_{n-1} approach the golden mean in a limiting sense. We have also shown that the ratio of adjacent Fibonacci p-numbers $F_p(n)/F_p(n-1)$ approaches in the limit some positive number τ_p that is a positive root of the polynomial equation [8]:
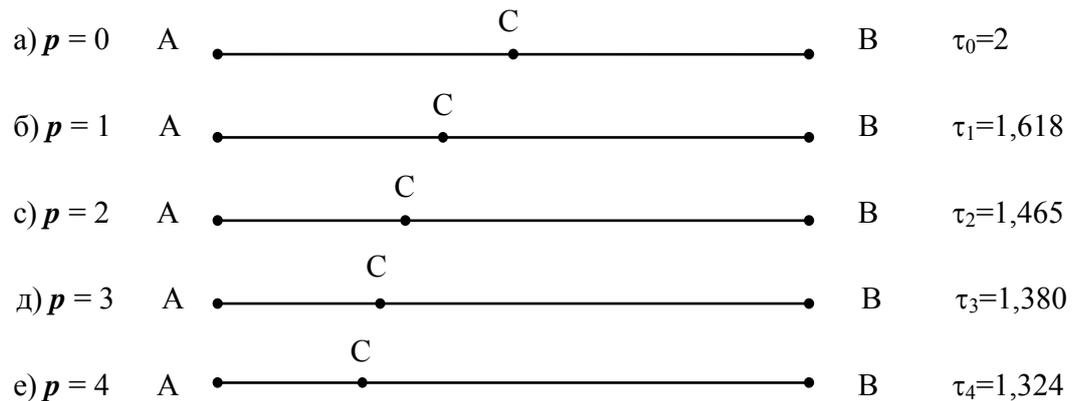
$$x^{p+1} = x^p + 1, \qquad (45)$$

where $p=0, 1, 2, 3, \ldots$.

The numbers τ_p are referred to as *golden p-proportions* since for the case $p=1$ the number τ_p coincides with the classical golden proportion. We call Equation (45) the *golden algebraic equation* because for the case $p=1$ Equation (45) reduces to Equation (1)[9].

Note that for the case $p=0$, $\tau_p = \tau_0 = 2$, and for $p=\infty$, $\tau_p = \tau_\infty = 1$. This means that on the number line between the numerical constants 1 and 2 there are an infinite number of numerical constants τ_p, which expresses some deep mathematical properties of Pascal triangle.

We have formulated the following geometric problem called a *problem of the golden p-section*. Consider a given integer $p=0, 1, 2, 3, \ldots$, and divide a line segment *AB* by a point *C* according to the following relation also shown in Fig. 7:

$$\frac{CB}{AC} = \left(\frac{AB}{CB}\right)^p \qquad (46)$$

а) **p = 0** A ———————————•C———————————• B $\tau_0 = 2$

б) **p = 1** A ——————•C——————————————• B $\tau_1 = 1{,}618$

с) **p = 2** A —————•C———————————————• B $\tau_2 = 1{,}465$

д) **p = 3** A ————•C————————————————• B $\tau_3 = 1{,}380$

е) **p = 4** A ———•C—————————————————• B $\tau_4 = 1{,}324$

Figure 7. The golden p-sections ($p = 0, 1, 2, 3, \ldots$)

A solution to problem (46) to find a section point C of line segment AB reduces to a search for a positive root of the Equation (45), i.e., the division of a line segment according to Equation (46) is equal to the golden p-proportion [8].

Consider two cases of the golden p-section: for the case $p=0$ the golden p-section reduces to a bisection (Fig. 7a), while for the case $p = 1$ it reduces to the classical golden section (Fig. 7b).

For the remaining values of p we get an infinite number of proportional divisions of the line segment according to Relation (46). In particular, it is easy to show that for the case $p \to \infty$, $\tau_p \to 1$.

The following identity relating powers of the golden p-proportion degrees follows directly from (45):

$$\tau_p^n = \tau_p^{n-1} + \tau_p^{n-p-1} = \tau_p \times \tau_p^{n-1} \qquad (47)$$

where n is an arbitrary integer.

Equation (45), being the equation for the roots of a $(p+1)$-th degree polynomial, has $p+1$ roots: x_1, x_2, x_3, …, x_p, x_{p+1}. Later we show that root x_1 always coincides with the golden p-proportion τ_p, i.e., $x_1 = \tau_p$. Also each of the roots x_k satisfies Equation 47,

$$x_k^n = x_k^{n-1} + x_k^{n-p-1} = x_k \times x_k^{n-1} \tag{48}$$

.

The roots of Equation (45) satisfy the following identities [9]:

$$x_1 + x_2 + x_3 + x_4 + \ldots + x_p + x_{p+1} = 1 \tag{49}$$
$$x_1 x_2 x_3 x_4 \ldots x_{p-1} x_p x_{p+1} = (-1)^p \tag{50}$$

Also, for $p=1, 2, 3, \ldots$ and $k=1, 2, 3, \ldots, p$ [9]:

$$x_1^k + x_2^k + x_3^k + x_4 + \ldots + x_p^k + x_{p+1}^k = 1. \tag{51}$$

We have proved that there are an infinite number of the golden algebraic equations that are derived from equation (45):

$$x^n = F_p(n-p+1)x^p + \sum_{t=0}^{p-1}\left[F_p(n-p-t)x^t\right] \tag{52}$$

where $n \geq p+1$.

Note that Equations (52) have a common root τ_p.

In particular, for the case $p=1$, Equation (52) takes the form:

$$x^n = F_n x + F_{n-1}, \tag{53}$$

where F_n, F_{n-1} are Fibonacci numbers.

For the case $n=4$ Equation (53) takes the form:

$$x^4 = 3x + 2. \tag{54}$$

Equation (54) leads to the unexpected result that it describes the energy levels of the molecule butadiene, a chemical used for rubber production. The famous American physicist, Richard Feynman, expressed his admiration for Equation (54) with the following remark: *"What miracles exist in mathematics! According to my theory, the Golden Proportion of the ancient Greeks gives the condition for the minimal energy of the butadiene molecule."*

This connection to chemistry raises our level of interest in the golden algebraic equations. Perhaps these equations could be used to model the energy conditions of other molecules.

6.3. The Generalized Principle of the Golden Section

Consider the main identity for the golden p-proportion given by (47), and divide all terms by τ_p to get the following identity for a "unit":

$$1 = \tau_p^0 = \tau_p^{-1} + \tau_p^{-p-1}. \tag{55}$$

From (47) and (55), we construct the following partition of unity by the golden p-proportion [13]:

$$
\begin{aligned}
1 = \tau_p^0 &= \tau_p^{-1} + \tau_p^{-(p+1)} \\
\tau_p^{-(p+1)} &= \tau_p^{-(p+1)-1} + \tau_p^{-2(p+1)} \\
\tau_p^{-2(p+1)} &= \tau_p^{-2(p+1)-1} + \tau_p^{-3(p+1)}
\end{aligned}
\tag{56}
$$

$$1 = \tau^0 = \tau_p^{-1} + \tau_p^{-(p+1)-1} + \tau_p^{-2(p+1)-1} + \tau_p^{-3(p+1)-1} + \quad \ldots \quad = \sum_{i=1}^{\infty} \tau_p^{-(i-1)(p+1)-1}$$

i.e.,

$$1 = \tau_p^{-1} + \tau_p^{-(p+1)} = \sum_{i=1}^{\infty} \tau_p^{-(i-1)(p+1)-1}, \tag{57}$$

Note that for $p=0$ Equation (57) takes the following form:

$$1 = 2^0 = 2^{-1} + 2^{-1} = \sum_{i=1}^{\infty} 2^{-i} \tag{58}$$

This partition of unity expresses the so-called *"Dichotomy Principle"* of Zeno the ancient Greek philosopher that is used widely in mathematics and science [29].

Finally, for $p=1$ identity (57) takes the following form:

$$1 = \tau^0 = \tau^{-1} + \tau^{-2} = \sum_{i=1}^{\infty} \tau^{-(2i-1)} \tag{59}$$

Note that identity (59) expresses the *Golden Section Principle* that also came from ancient science.

Thus, identity (57) defines a more general principle for dividing a line segment into two parts, referred to in [13] as the *"Generalized Principle of the Golden Section"* for which the "Dichotomy Principle" and the "Golden Section Principle" are sub-cases.

6.4. Generalized Binet formulas and the continuous functions for the Fibonacci and Lucas p-numbers

Generalized Binet formulas enable Fibonacci p-numbers to be expressed in analytic form [10]. For a given integer $p>0$ any Fibonacci p-number $F_p(n)$ for integer n can be represented in terms of the roots of the golden algebraic Equation (46) in the following form:

$$F_p(n) = k_1(x_1)^n + k_2(x_2)^n + \ldots + k_{p+1}(x_{p+1})^n \tag{60}$$

where $k_1, k_2, \ldots, k_{p+1}$ are constant coefficients that are solutions of the following system of algebraic equations:

$$F_p(0) = k_1 + k_2 + \ldots + k_{p+1} = 0$$
$$F_p(1) = k_1 x_1 + k_2 x_2 + \ldots + k_{p+1} x_{p+1} = 1$$
$$F_p(2) = k_1(x_1)^2 + k_2(x_2)^2 + \ldots + k_{p+1}(x_{p+1})^2 = 1 \tag{61}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$F_p(p) = k_1(x_1)^p + k_2(x_2)^p + \ldots + k_{p+1}(x_{p+1})^p = 1$$

In particular, for $p=1$ the Fibonacci p-numbers reduce to the classical Fibonacci numbers F_n. For this special case expression (60) reduces to the well-known expression [2, 3]:

$$F_n = k_1(x_1)^n + k_2(x_2)^n, \tag{62}$$

where x_1, x_2 are the roots of Equation (1):

$$x_1 = \tau = \frac{1+\sqrt{5}}{2}; x_2 = -\frac{1}{\tau}, \tag{63}$$

and coefficients k_1, k_2 are solutions of the system of equations:

$$\begin{cases} F_0 = k_1 + k_2 = 0 \\ F_1) = k_1 \tau + k_2(-1/\tau) = 1 \end{cases} \tag{64}$$

where,

$$k_1 = \frac{1}{\sqrt{5}}; \; k_2 = -\frac{1}{\sqrt{5}}. \tag{65}$$

If we substitute (63) and (65) into Equation (62), we get the Binet formula for Fibonacci numbers:

$$F_n = \frac{\tau^n - \left(-\frac{1}{\tau}\right)^n}{\sqrt{5}}. \tag{66}$$

It has been shown that Lucas numbers L_n can be expressed in terms of the roots x_1, x_2 in the following form [2,3]:

$$L_n = x_1^n + x_2^n = \tau^n + \left(-\frac{1}{\tau}\right)^n \tag{67}$$

We have shown [10] that the sum of the n-th powers of the roots $x_1, x_2, \ldots, x_{p+1}$ of the Equation (45) establishes a new class of recursive sequences referred as *Lucas p-numbers $L_p(n)$, where $p=0, 1, 2, 3, \ldots$, i.e.,

$$L_p(n) = (x_1)^n + (x_2)^n + \ldots + (x_{p+1})^n \tag{68}$$

Using properties (48)-(51) of Equation (46) series (68) follows from the recursive relation:

$$L_p(n) = L_p(n-1) + L_p(n-p-1) \tag{69}$$

with the following initial terms:

$$L_p(0) = p+1 \tag{70}$$
$$L_p(1) = L_p(2) = \dots = L_p(p) = 1. \tag{71}$$

Table 4 gives the values of the Lucas p-numbers for the cases $p=1, 2, 3, 4$.

Table 4. Lucas p-numbers

N	0	1	2	3	4	5	6	7	8	9	10	11	12
$L_1(n)$	2	1	3	4	7	11	18	29	47	76	123	199	322
$L_2(n)$	3	1	1	4	5	6	10	15	21	31	46	67	98
$L_3(n)$	4	1	1	1	5	6	7	8	13	19	26	34	47
$L_4(n)$	5	1	1	1	1	6	7	8	9	10	16	23	31

Thus, the main result of reference [10] is a generalization of the Binet formulas and the development of a new class of recursive numerical sequences, the *generalized Lucas numbers* or *Lucas p-numbers,* given either in analytical form (68) or in recursive form (69)-(71). These new recursive numerical sequences are of great theoretical interest, and they may prove useful for the characterization of certain natural processes.

As shown above, hyperbolic Fibonacci and Lucas functions (13)-(16) and (18)-(21) are generalizations of the Binet formulas (9)-(10) to the continuous domain [11]. The generalized Binet formulas (60), (68), which allow us to express the Fibonacci and Lucas p-numbers in terms of the roots of Equation (45), can be used to define continuous functions for the Fibonacci and Lucas p-numbers as was done for hyperbolic Fibonacci and Lucas functions. The equations for the continuous functions of the Fibonacci and Lucas p-numbers are more complex than Equations (13)-(16) and (18)-(21). Nevertheless, the results of article [11] can be considered to be a variant of the golden hyperbolic models of Nature with the potential to be useful in the natural sciences.

6.5. Fibonacci matrices

Over past few decades, the theory of Fibonacci numbers has been complemented by the theory of the *Fibonacci Q-matrix* [7]. The latter is a square 2×2 matrix of the following form:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{72}$$

Note that the determinant of the Q-matrix is equal to -1, i.e.,

$$\text{Det } Q = -1. \tag{73}$$

The n-th power of the Q-matrix can be expressed as follows:

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \tag{74}$$

where n is an arbitrary integer and F_{n-1}, F_n, F_{n+1} are Fibonacci numbers [7].

Using (73), it is easy to get the following expression for the determinant of matrix (74):

$$\text{Det } Q^n = (-1)^n, \tag{75}$$

where *n is an integer.*

On the other hand, the determinant of matrix (74) is given by,

$$Det\ Q^n = F_{n+1}F_{n-1} - F_n^2 = (-1)^n \tag{76}$$

Identity (76) is known as the ***"Cassini formula"*** in honor of the well-known 17-th century astronomer Giovanni Cassini (1625-1712) who derived this formula [26]. It follows that the Q-matrix is an expression of the key Identity (76) for Fibonacci numbers, and that fundamental property (75) can be considered to be a compact form of the "Cassini formula"!

The idea of the Q-matrix was developed in [12], where the following square $(p+1)\times(p+1)$, Q_p-*matrix* was introduced:

$$Q_p = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{77}$$

for $p = 0, 1, 2, 3, \ldots$.

It contains a $p \times p$-*identity* matrix bordered by the last row, which consists of 0's and a leading 1, and the first column, which consists of 0's embraced by a pair of 1's. We list the Q_p-matrices for $p = 0, 1, 2, 3, 4$:

$$Q_0 = (1) ; \qquad Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = Q ; \qquad Q_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} ;$$

$$Q_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} ; \qquad Q_4 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

The main result of reference [12] is a proof of the following expression for the n-th power of the Q_p-matrix:

$$Q_p^n = \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix} \tag{78}$$

where $p = 0, 1, 2, 3, \ldots, n = 0, \pm 1, \pm 2, \pm 3, \ldots$, and elements of the matrix are the Fibonacci p-numbers.

Note that the class of matrices (77) has an interesting recursive property. If we cross out the last, that is, $(p+1)$-th column and the next to the last, that is, p-th row in the Q_p-matrix, then the matrix reduces to matrix Q_{p-1}. This means that the determinant of the Q_p-matrix differs from the determinant of the Q_{p-1}-matrix only by its sign, i.e.,

$$\text{Det } Q_{p-1} = -\text{Det } Q_p . \tag{79}$$

Taking into consideration (73) and (79),

$$\text{Det } Q_p = (-1)^p. \tag{80}$$

And it follows that,

$$\text{Det } Q_p^n = (-1)^{pn}. \tag{81}$$

where $p = 0, 1, 2, 3, \ldots$; $n = 0, \pm 1, \pm 2, \pm 3, \ldots$.

The main result of reference [12] is the development of a new class of square matrices satisfying Equation (81).

It is clear that matrices (77) and (78) can be used for the future progress of Fibonacci research. Here expression (81) can be considered to be a generalization of the "Cassini formula" . For example, for the case $p=2$ the generalized "Cassini formula" takes the form:

$$\text{Det}\, Q_2^n = F_2(n+1)[F_2(n-2)F_2(n-2) - F_2(n-1)F_2(n-3)] + F_2(n)[F_2(n)F_2(n-3) - F_2(n-1)F_2(n-2)] +$$
$$+ F_2(n-1)[F_2(n-1)F_2(n-1) - F_2(n)F_2(n-2)] = 1. \tag{82}$$

While the Cassini formula gives a connection between three adjacent Fibonacci numbers, formula (82) connects five adjacent Fibonacci 2-numbers $F_2(n-3)$, $F_2(n-2)$, $F_2(n-1)$, $F_2(n)$ and $F_2(n+1)$ for any integer value of n.

There are an infinite number of generalized Cassini formulas similar to (82) for $p=1, 2, 3, \ldots$

6.6. The golden matrices

Stakhov has introduced a class of *golden matrices [13]*. Let us represent matrix (74) in the form of two matrices, one for even ($n=2k$) and the other for odd ($n=2k+1$) values of n:

$$Q^{2k} = \begin{pmatrix} F_{2k+1} & F_{2k} \\ F_{2k} & F_{2k--1} \end{pmatrix} \tag{83}$$

$$Q^{2k+1} = \begin{pmatrix} F_{2k+2} & F_{2k+1} \\ F_{2k+1} & F_{2k} \end{pmatrix} \tag{84}$$

Using relation (22), write the matrices (83), (84) in terms of symmetric hyperbolic Fibonacci functions (18), (19):

$$Q^{2k} = \begin{pmatrix} cFs(2k+1) & sFs(2k) \\ sFs(2k) & cFs(2k-1) \end{pmatrix} \tag{85}$$

$$Q^{2k+1} = \begin{pmatrix} sFs(2k+2) & cFs(2k+1) \\ cFs(2k+1) & sFs(2k) \end{pmatrix} \tag{86}$$

where k is an integer.

Substituting discrete variable k in the matrices (85), (86) with continuous variable x, results in two unusual matrices that are functions of x:

$$Q^{2x} = \begin{pmatrix} cFs(2x+1) & sFs(2x) \\ sFs(2x) & cFs(2x-1) \end{pmatrix} \tag{87}$$

$$Q^{2x+1} = \begin{pmatrix} sFs(2x+2) & cFs(2x+1) \\ cFs(2x+1) & sFs(2x) \end{pmatrix} \tag{88}$$

It is clear that Matrices (87) and (88) are generalizations of the Q-matrix (74) for the continuous domain. These matrices have a number of unusual mathematical properties. For example, for when $x = \dfrac{1}{4}$ matrix (87) takes the following form:

$$Q^{\frac{1}{2}} = \sqrt{Q} = \begin{pmatrix} cFs(\dfrac{3}{2}) & sFs(\dfrac{1}{2}) \\ sFs(\dfrac{1}{2}) & cFs(-\dfrac{1}{2}) \end{pmatrix} \tag{89}$$

It is difficult to imagine what is meant by the "square root" of the Q-matrix», but just such a "Fibonacci fantasy" follows from Equation (89)!

Using properties (24) and (25) of the symmetric hyperbolic functions to compute the determinants of matrices (87) and (88) leads to the result,

$$\text{Det } Q^{2x} = 1 \tag{90}$$
$$\text{Det } Q^{2x+1} = -1 \tag{91}$$

Therefore these determinants are independent of x and equal identically to 1 in absolute value. In fact identities (90) and (91) are generalizations of the Cassini formula to the continuous domain!

At present it is difficult to predict where matrices (78), (87), (88), having unusual mathematical properties (79), (81),(90), (91), might be used. However, they were recently applied to coding theory and cryptography [13, 24, 25].

4. Conclusion

References [4-25] have been successful in developing new mathematical equipment, based on both the classical golden section and the golden p-sections, useful for describing the "hyperbolic worlds" of Nature. The most important of these golden mathematical models are the following:

1. Hyperbolic Fibonacci and Lucas functions. A consequence of their introduction has been the realization that the classical hyperbolic functions (12), (13), which are useful in mathematics and theoretical physics, are not the only tools for creating mathematical models of the "hyperbolic world". In addition to models based on classical hyperbolic functions (Lobachevsky's hyperbolic geometry, Minkowsky's geometry, etc.), there is a golden hyperbolic world based on hyperbolic Fibonacci and Lucas functions [4-7]. **The golden hyperbolic world exists objectively and independently of our awareness. This "hyperbolic world" is ubiquitous to Living Nature. In particular, it appears in pinecones, sunflower heads, pineapples, cacti, and in florescence's of various flowers in the form of the Fibonacci and Lucas spirals on the surfaces of these biological objects (the phyllotaxis law). Note that the hyperbolic Fibonacci and Lucas functions [4-7], which underlie phyllotaxis phenomena are not mere inventions of Fibonacci-mathematicians, but rather they objectively reflect the mathematical laws underlying the geometry of Living Nature.**

2. The Golden Shofar. The Golden Shofar is another new model of the golden hyperbolic world. The article **«Hyperbolic Universes with a Horned Topology and the CMB Anisotropy»** [28] proved a great surprise to the authors because it showed that **the geometry of the Universe is quite possibly is "shofarable" by its structure**.

3. The Generalized Principle of the Golden Section, which is based on the golden p-proportions**,** is of general scientific interest and could influence the development of all natural sciences wherever the "Dichotomy Principle" and the classical Golden Section Principle are used [29].

In summary, the authors would like to express their surprise that it has taken so long for mathematicians and physicists to give proper attention to the development of the mathematical needed to model the golden hyperbolic world. However, toward the end of the 20th century, due to the efforts of several theoretical physicists, the situation began to change. References [30-39] demonstrate a substantial interest on the part of modern theoretical physicists in the golden section and the golden hyperbolic world. The works of Shechtman, Butusov, Mauldin and William, El Naschie, Vladimirov, Petoukhov and other scientists show that it is impossible to imagine future progress in physical and cosmological research without a consideration of the golden section**.** In this connection, the golden genomatrices of the Russian scientist Sergey Petoukhov [39] are of great importance for the future development of science. **Petoukhov's discovery shows that the golden section underlies the genetic code!**

It seems that the dramatic history of the golden section, which has continued over several millennia, may conclude with a great triumph for the golden section in the beginning of the 21st century, the "Century of Harmony." Penrose tiles, a resonance theory of the Solar system (Molchanov and Butusov), quasi-crystals (Shechtaman), fullerenes (Kroto, Curl and Smalley, 1996 Nobel Prize in chemistry) were only precursors of this triumph. "Harmony Mathematics" (Stakhov), hyperbolic Fibonacci and Lucas functions (Stakhov, Tkachenko, Rozin), Bodnar's geometry, the "Law of structural harmony of systems" (Soroko), El Nashie's theory of E-infinity, Fibonacci matrices and a new coding theory (Stakhov), the golden matrices and a new cryptography (Stakhov), and finally the golden genomatrices (Petoukhov) are only a partial list of recent scientific discoveries based on the golden section. These discoveries give reason to suppose that **the golden**

section may be a kind of "metaphysical knowledge", "pre-number", or "universal code of Nature", which could become the foundation for the future development of science, in particular, mathematics, theoretical physics, genetics, and computer science.

In 1996, The Fibonacci Quarterly, published the paper "On Fibonacci hyperbolic trigonometry and modified numerical triangles" written by the well-known Fibonacci-mathematician Zdzislaw W. Trzaska [40]. On the one hand, interest of Fibonacci-mathematicians in hyperbolic Fibonacci functions should be encouraged, on the other hand a comparison of Trzaska's paper [40] and Stakhov and Tkachenko's paper [4] showed that, in the part concerning hyperbolic Fibonacci functions, Trzaska used all of the material of Stakhov and Tkachenko's paper [4] without attribution thus creating the perception that he was the original discoverer of the hyperbolic Fibonacci functions. However, since reference [4] was published in 1993, three years before the publication of Trzaska's paper, this establishes Stakhov and Tkachenko's priority in the discovery of hyperbolic Fibonacci and Lucas functions.

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