GENERALIZED GENOMIC MATRICES, SILVER MEANS, AND PYTHAGOREAN TRIPLES

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ABSTRACT

Petoukhov has shown that a family of bisymmetric 2^n x 2^n matrices encode the structure of the four RNA and DNA bases and 64 codons that make up the 20 amino acids in all living structures. He discovered that the elements of the square roots of these matrices are all powers of the golden mean. We have generalized his matrices and shown that the square roots of general bisymmetric matrices are generalizations of the golden mean including a subclass that correspond to the family of silver means. Powers of these matrices are also shown to generate all Pythagorean triples. The integers in these matrices are identical to the set of integers in a table attributed to the second century Syrian mathematician, Nicomachus, who used them to describe the ancient musical scale of Pythagoras.

Keywords: dna/rna, golden mean, amino acids, Nichomachus

1. INTRODUCTION

Petoukhov (2001, 2004), (He, 2005) has studied a family of bisymmetric 2^n x 2^n matrices that code the structure of the four DNA/RNA bases, the 64 codons that make up the 20 amino acids in all living structures, and beyond that, the proteins assembled from the amino acids as building blocks. As the result of his studies he has found that the amino acids express certain degeneracies, 8 with high degeneracy containing 4 or more codons, and 12 with low degeneracy, containing less than 4 codons. These degeneracies are propagated through the 17 different genome classes of RNA/DNA. The particular class of DNA/RNA that we will be studying in this paper is the class of mitochondrial DNA. Although different groups of codons correspond to the same amino acid in different genome classes, the quality of the degeneracy (high or low) is preserved. The first matrix of the family expresses the fact that two of the RNA bases have 3 hydrogen bonds while the other two have 2 hydrogen bonds. The elements of the rows and columns of this family of matrices reproduce the sequences of musical fifths, i.e., integer ratios of 3:2, found in a table attributed to the Syrian mathematician of the first and second century AD, Nicomachus (Kappraff, 2000a). The integer values in this table have multiplicities given by the rows of Pascal's triangle. The square roots of this family of matrices have entries that are all powers of the golden mean. A brief discussion of Petoukhov's approach to genetic coding is given in Appendix A.

We have generalized Petoukhov's matrices to a family of bisymmetric matrices in which the first 2x2 matrix has a pair of positive real numbers as elements, but are otherwise arbitrary. Bisymmetric matrices are matrices whose elements are symmetric with respect to both left and right leaning diagonals. We derive general formulas for the elements of the square root of this matrix. They are irrational numbers that are generalizations of the golden mean. In fact for a subclass of the matrices the elements of the square root matrix are generalizations of the golden mean known as silver means. Finally, we show that when the elements of the bisymmetric matrix are positive integers, powers of these matrices generate Pythagorean triples.

2. PETOUKHOV'S GENOMIC MATRICES

Petoukhov has shown that the four nitrogenous bases that make up RNA and DNA, adenine, cytosine, guanine, and uracil/thymine: A,C,G,U/T are equivalent in two different ways.

a.. C=U and A=G according to the relation, "pyrimidine or purine."
b. C=G and A=U/T according to the relation, "possesses three hydrogen bonds or two hydrogen bonds (Watson, 1953)"

These two properties characterize a family of matrices related to the four bases. The first of these matrices, the 2x2 RNA Matrix 1, specifies the four bases in which C is coded by $\underline{1}1$, A by $\underline{1}0$, U by $\underline{0}1$ and G by $\underline{0}0$. For relation a) C and U are pyrimidines and are assigned the value 1 in column 1 while A and G are purines and are assigned the value 0 in column 2 . For relation b) C and G have three hydrogen bonds coded by $\underline{1}1$ and $\underline{0}0$ respectively in Matrix 1, while A and U have two hydrogen bonds coded by $\underline{1}0$, and $\underline{0}1$ respectively. In this manner the bases are assigned the values 3 and 2 respectively along the two diagonals of Matrix M_1 shown below.

$$\begin{array}{cc} & \begin{array}{cc} 1 & 0 \end{array} \\ \begin{array}{c} \underline{1} \\ \underline{0} \end{array} & \begin{bmatrix} C & A \\ U & G \end{bmatrix} \end{array} \tag{1}$$

$$M_1 = \begin{array}{cc} & \begin{array}{cc} 1 & 0 \end{array} \\ \begin{array}{c} \underline{1} \\ \underline{0} \end{array} & \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \end{array}.$$

Matrix 2 shows that the bit strings that code the four bases are organized according to binary reflecting Gray code values of the numbers in parentheses.

$$M_{coord} = \begin{bmatrix} \underline{1}1(2) & \underline{1}0(3) \\ \underline{0}1(1) & \underline{0}0(0) \end{bmatrix} \tag{2}$$

To obtain the Gray code value of an integer start with 0 equal 0 in Gray code. Then take the Gray code value of an integer and change 1 to 0 or 0 to 1 in the least significant place so long as it does not reproduce a Gray code value already obtained to obtain Gray code for the next integer. For example, 1 is represented by 1 (or 01) in Gray code, but 2 changes 01 to 11. Notice that adjacent values in Matrix 2, including wrap-around, differ by a single bit. We used Gray code because the amino acids organize themselves so that a mutation of a single base within a codon would still preserve the identity of the amino acid, i.e., indices of adjacent codons within a single amino acid should differ by no more that a single digit as you can see by examining the amino acids in Fig. 4a and b. The codons for any amino acid lie in adjacent rows and columns and therefore, their Gray code representations differ by a single bit. Petoukhov organized these matrices using binary and got essentially similar results which is remarkable since adjacent binary indices need not differ by a single code digit, yet the amino acids organize themselves in binary in just this manner.

The 4x4 matrix M_2 is shown below. The rows and columns are numbered from bottom to top and right to left by binary reflecting Gray code.

$$\begin{array}{c c}
 & \begin{array}{cccc} 10 & 11 & 01 & 00 \end{array} \\
\begin{array}{c} \underline{10} \\ \underline{11} \\ \underline{01} \\ \underline{00} \end{array} & \begin{bmatrix} CG & CU & AU & AG \\ CA & CC & AC & AA \\ UA & UC & GC & GA \\ UG & UU & GU & GG \end{bmatrix}
\end{array}
\qquad
\mathrm{M}_2 = \begin{array}{c c}
 & \begin{array}{cccc} 10 & 11 & 01 & 00 \end{array} \\
\begin{array}{c} \underline{10} \\ \underline{11} \\ \underline{01} \\ \underline{00} \end{array} & \begin{bmatrix} 9 & 6 & 4 & 6 \\ 6 & 9 & 6 & 4 \\ 4 & 6 & 9 & 6 \\ 6 & 4 & 6 & 9 \end{bmatrix}
\end{array}
\qquad (3a,b)$$

and the indices are organized again by binary reflecting Gray code,

$$\mathrm{M}_{coord} = \begin{bmatrix} \underline{10}10(12) & \underline{10}11(13) & \underline{10}01(14) & \underline{10}00(15) \\ \underline{11}10(11) & \underline{11}11(10) & \underline{11}01(9) & \underline{11}00(8) \\ \underline{01}10(4) & \underline{01}11(5) & \underline{01}01(6) & \underline{01}00(7) \\ \underline{00}10(3) & \underline{00}11(2) & \underline{00}01(1) & \underline{00}00(0) \end{bmatrix}$$

where $\underline{10}10$, i.e., $\underline{1}1,\underline{0}0$ corresponds to CG and so we assign it the product of the corresponding hydrogen bonds, 3x3=9. Likewise $\underline{10}11$, i.e., $\underline{1}1$, $\underline{0}1$, corresponds to CU and so is assigned the value 3x2=6, etc. These numbers correspond to the number of ways that the hydrogen bonds can interact between bases. When represented in binary as Petoukhov did, $M_2 = M_1 \otimes M_1$ where $\otimes$ is the symbol for tensor product. Likewise M_n is represented by a tensor exponentiation of the nth degree. Continuing with our Gray code representation, M_3 results in the 8x8 matrix,

$M_3 =$

$$\begin{bmatrix} CGG(Arg) & CGU(Arg) & CUU(Leu) & CUG(Leu) & AUG(Met) & AUU(Ile) & AGU(Ser) & AGG(Stop) \\ CGA(Arg) & CGC(Arg) & CUC(Leu) & CUA(Leu) & AUA(Met) & AUC(Ile) & AGC(Ser) & AGA(Stop) \\ CAA(G\ln) & CAC(His) & CCC(\Pr o) & CCA(pro) & ACA(Thr) & ACC(Thr) & AAC(Asn) & AAA(Lys) \\ CAG(G\ln) & CAU(His) & CCU(\Pr o) & CCG(\Pr o) & ACG(Thr) & ACU(Thr) & AAU(Asn) & AAG(Lys) \\ UAG(Stop) & UAU(Tyr) & UCU(Ser) & UCG(Ser) & GCG(Ala) & GCU(Ala) & GAU(Asp) & GAG(Glu) \\ UAA(Stop) & UAC(Tyr) & UCC(Ser) & UCA(Ser) & GCA(Ala) & GCC(Ala) & GAC(Asp) & GAA(Glu) \\ UGA(Trp) & UGC(Cys) & UUC(Phe) & UUA(Leu) & GUA(Val) & GUC(Val) & GGC(Gly) & GGA(Gly) \\ UGG(Trp) & UGU(Cys) & UUU(Phe) & UUG(Leu) & GUG(Val) & GUU(Val) & GGU(Gly) & GGG(Gly) \end{bmatrix}$$

4a)

and,

$$M_3 = \begin{array}{c} \\ \underline{100} \\ \underline{101} \\ \underline{111} \\ \underline{110} \\ \underline{010} \\ \underline{011} \\ \underline{001} \\ \underline{000} \end{array} \begin{array}{c} \begin{array}{cccccccc} 100 & 101 & 111 & 110 & 010 & 011 & 001 & 000 \end{array} \\ \begin{bmatrix} 27 & 18 & 12 & 18 & 12 & 8 & 12 & 18 \\ 18 & 27 & 18 & 12 & 8 & 12 & 18 & 12 \\ 12 & 18 & 27 & 18 & 12 & 18 & 12 & 8 \\ 18 & 12 & 18 & 27 & 18 & 12 & 8 & 12 \\ 12 & 8 & 12 & 18 & 27 & 18 & 12 & 18 \\ 8 & 12 & 18 & 12 & 18 & 27 & 18 & 12 \\ 12 & 18 & 12 & 8 & 12 & 18 & 27 & 18 \\ 18 & 12 & 8 & 12 & 18 & 12 & 18 & 27 \end{bmatrix} \end{array} \qquad (4b)$$

where $\underline{100}100$, i.e., $\underline{1}1$, $\underline{0}0$, $\underline{0}0$ corresponds to CGG so we assign it the product 3x3x3= 27, $\underline{111}100$, i.e., $\underline{1}1$, $\underline{1}0$, $\underline{1}0$ corresponds to CAA so we assign the product 3x2x2 = 12, etc. In Matrix 4a the 20 amino acids are listed in parentheses. Notice that there are 8 amino acids with high-degeneracy (4 or more codons) and 12 with low-degeneracy (less than 4 codons). If the high degeneracy codons in Matrix 4a are shaded gray the resulting pattern has 2-fold rotational symmetry as shown in Fig. 1, while if all matrix locations in Matrix 4b of a given integer value are coded by a different color, the resulting pattern has the symmetry of D_2, symmetric in both diagonals as shown in Fig. 2a. When the elements of the matrix are ordered according to binary, as Petoukhov did, the result is shown in Fig. 2b, also with D_2 symmetry.. The designs in Fig. 2a and b inspired the American quilter, Elaine Ellison to create the lovely quilts shown in Fig. 2c which she named, “The music of the Genes” for reasons that will be described below. Petoukhov has shown that while there are 17 known genome classes of DNA/RNA, if a codon codes for one amino acid in one class of DNA but another amino acid in a different DNA/RNA class, the degeneracy will be preserved so that the pattern of Fig. 1 is invariant over all 17 genome classes of DNA/RNA (Petoukhov, 2005). Also note that bit strings of adjacent codons that make up an amino acid necessarily differ by a single bit because of the nature of binary reflecting Gray code

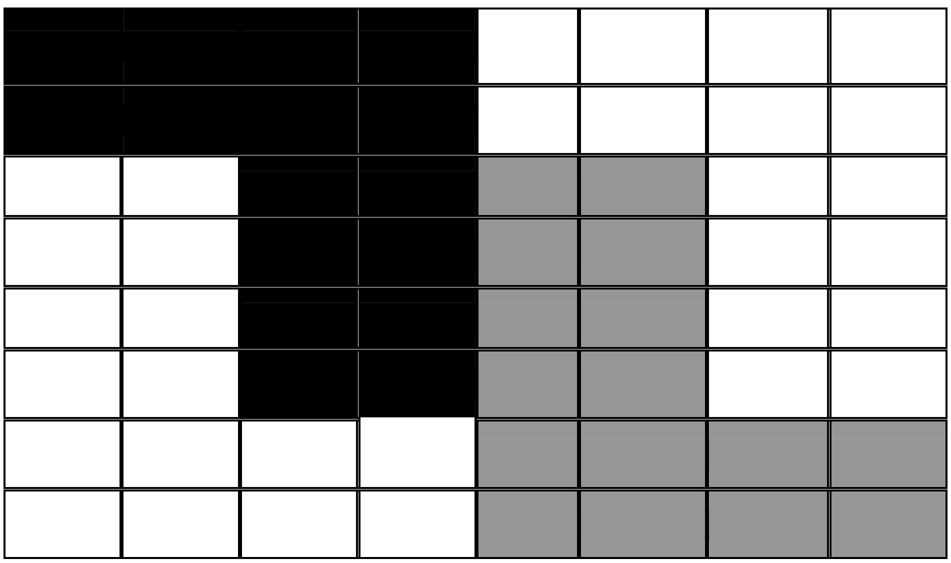


Figure 1. High and low degeneracy amino acids.

27	18	12	18	12	8	12	18
18	27	18	12	8	12	18	12
12	18	27	18	12	18	12	8
18	12	18	27	18	12	8	12
12	8	12	18	27	18	12	18
8	12	18	12	18	27	18	12
12	18	12	8	12	18	27	18
18	12	8	12	18	12	18	27

Figure 2a. RNA matrix of amino acids – Gray Code ordering

27	18	18	12	18	12	12	8
18	27	12	18	12	18	8	12
18	12	27	18	12	8	18	12
12	18	18	27	8	12	12	18
18	12	12	8	27	18	18	12
12	18	8	12	18	27	12	18
12	8	18	12	18	12	27	18
8	12	12	18	12	18	18	27

Figure 2b. RNA matrix of amino acids – Binary ordering

Figure 2c. "The Music of the Genes". A quilt pattern by Elaine Ellison.

Notice that in M_2 the natural numbers 4,6,9 appear while in M_3 the natural numbers 8, 12,18, 27 appear, with each row and column having the same sequence of positive integers with no integer appearing adjacent to itself in a row or column. These sequences come from a triangle of numbers attributed to the 2nd century AD Syrian mathematician Nicomachus (Kappraff, 2000) and represent successive sequences of musical fifths. The Nicomachus Triangle, T(n,k), is reproduced in Table 1 where the integers in the n-th row are $\{2^{n-k}3^k, 0 \le k \le n\}; n \ge 0$. Here if the central integer 6 is thought to be the length of a string representing a fundamental tone, then 4 and 9 of row 3 represent the string lengths corresponding to rising and falling musical fifths, ratios of 2:3 and 3:2. Also the fifth row represents the string lengths that give rise to a pentatonic scale with fundamental string length of 36 units while the integers in row 7 represent string lengths of a heptatonic scale with216 as the string length of the fundamental. The Triangle T(n,k) in Table 1 has the property that every row, column, diagonal, and line joining any two elements contains a nontrivial geometric progression.

Table 1. The Nicomachus Triangle, T(n,k)

1						
2	3					
4	6	9				
8	12	18	27			
16	24	36	54	81		
32	48	72	108	162	243	
64	96	144	216	324	486	729
			etc.			

Table 2. Pascal's Triangle

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1
		etc.		

T(n,k) is the triangle of coefficients in the expansion of $(2 + 3x)^n$; given by the generating function $\frac{1}{1-y(2+3x)}$. For example, 8,12,18,27 are generated by $(2+3x)^3 = 8+3\times 12x+3\times 18x^2+27x^3$ where we see that there is one 8, one 27, three 12's and three 18's in each row or column of matrix M_3. Furthermore if we set x = 1 we find the sum of the elements in each row or column of M_n equals 5^n. For M_3 the sum = 125. In other words, successive integers from a row of the Nicomachus triangle are multiplied by successive integers from rows of Pascal's Triangle, given in Table 2, e.g., $(1, 3, 3, 1) \bullet (8, 12, 18, 27)$ where $\bullet$ denotes dot product in order to sum the row and column elements of M_3.

Petoukhov has shown that,

$$P_1 = M_1^{1/2} = \begin{bmatrix} \tau & 1/\tau \\ 1/\tau & \tau \end{bmatrix} \tag{6}$$

where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden mean. Associating τ with 11 and 00, and $1/\tau$ with 10 and 01, we obtain matrices of the square roots of each of the higher M_n matrices denoted by P_n. For example,

$$P_2 = M_2^{1/2} = \begin{bmatrix} \tau^2 & 1 & 1/\tau^2 & 1 \\ 1 & \tau^2 & 1 & 1/\tau^2 \\ 1/\tau^2 & 1 & \tau^2 & 1 \\ 1 & 1/\tau^2 & 1 & \tau^2 \end{bmatrix} \qquad (7)$$

$\underline{10}10$ corresponds to $\tau \times \tau = \tau^2$, $\underline{11}00$ corresponds to $1/\tau \times 1/\tau = 1/\tau^2$, and $\underline{10}11$ corresponds to $\tau \times 1/\tau = 1$

3. GENERALIZED BISYMMETRIC MATRICES

We have been successful in generalizing Petoukhov's RNA matrices to bisymmetric matrices of the form

$$\mathrm{M}_1 = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \qquad (8a)$$

The higher order matrices M_n are determined in a similar manner as was done for the matrix with a = 3 and b=2. They contain columns and rows with integers from each column of the generalized Nicomachus Triangle in Table 3 with multiplicities given by Pascal's Triangle. For example, analogous to Matrix 3b, using elements of row 3 of Table 3,

$$\mathrm{M}_2 = \begin{bmatrix} a^2 & ab & b^2 & ab \\ ab & a^2 & ab & b^2 \\ b^2 & ab & a^2 & ab \\ ab & b^2 & ab & a^2 \end{bmatrix} \qquad (8b)$$

Table 3. Generalized Nicomachus Triangle

$$\begin{array}{lllll} 1 \\ b & a \\ b^2 & ab & a^2 \\ b^3 & ab^2 & a^2b & a^3 \\ b^4 & ab^3 & a^2b^2 & a^3b & a^4 \end{array} \qquad (9)$$

The elements of M_n are generated by $(b + ax)^n$ with each row and column of M_n summing to $(a+b)^n$. On the other hand in Sections 4, 5, and Appendix B we show that the square root of the M_1 and M_2 matrices can be expressed as,

$$P_1 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} \alpha^2 & \alpha\beta & \beta^2 & \alpha\beta \\ \alpha\beta & \alpha^2 & \alpha\beta & \beta^2 \\ \beta^2 & \alpha\beta & \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 & \alpha\beta & \alpha^2 \end{bmatrix} \tag{10a,b}$$

where,

$$\alpha = \frac{b}{2\beta} \quad \text{and} \quad \beta = \sqrt{\frac{a - \sqrt{a^2 - b^2}}{2}} \ . \tag{11a,b}$$

Therefore α and β are real for a>b and complex for a<b. Since $P_1^2 = M_1$, it follows that,

$$a = \alpha^2 + \beta^2, \ \ b = 2\alpha\beta \ , \tag{12a,b}$$

and from this it follows that,

$$\alpha + \beta = \sqrt{a+b}\ , \text{ and } \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{2a}{b} \tag{13a,b}$$

Also, α and β are roots of the fourth degree polynomial,

$$x^4 - (\alpha^2 + \beta^2)x^2 + \alpha^2\beta^2 = 0 \tag{14}$$

Making use of Equations 11a,b, Equation 14 is rewritten,

$$x^4 - ax^2 + \frac{b^2}{4} = 0 \tag{15}$$

where using Equation 11a and 12a,

$$a = \frac{b^2}{4\beta^2} + \beta^2 \tag{16}$$

Equation 15 can also be rewritten as,

$$x^4 = ax^2 - \frac{b^2}{4}\ , \tag{17}$$

and if we consider the geometric sequence,

$$1,\ \alpha^2,\ \alpha^4,\ \alpha^6,\ \alpha^8,\ \ldots \tag{18}$$

where ,

$$\alpha^{2n} = a\alpha^{2n-2} - \frac{b^2}{4}\alpha^{2n-4}, \tag{19}$$

this corresponds to a generalized “Fibonacci” sequence, $\{c_n\}$ in which,

$$c_1 = 0, c_2 = 1 \text{ and } c_n = ac_{n-1} - \frac{b^2}{4}c_{n-2}\ . \tag{20a}$$

The ratio of successive terms, $\frac{c_{n+1}}{c_n}$ approaches α^2 in the limit where, in the case that α is complex, then α^2 denotes the square of the absolute value. Also $\frac{c_{n+1}}{c_n}$ approaches α^2 from below.

If we let $g_n = c_n^2 - c_{k-1}c_{k-2}$, then g_n can be shown to satisfy the recursion,

$$g_n = a^2 g_{n-1} + \left(\frac{b^4}{16} - \frac{a^2b^2}{4}\right)g_{n-2}. \tag{20b}$$

Setting b = 2 and letting $a = N^2 \pm 2$, Equation 16 reduces to,

$$\frac{1}{\beta^2} + \beta^2 = N^2 \pm 2 \quad \text{and} \quad \alpha = 1/\beta\,. \tag{21a,b}$$

.
It follows that,

$$\frac{1}{\beta}\ \mu\ \beta = N \tag{22a}$$

Since $\alpha = 1/\beta$, Equation 22a is rewritten,

$$\alpha\ \mu\ \frac{1}{\alpha} = N \tag{22b}$$

We refer to solutions of the equations,

$$x - 1/x = N \quad \text{and } x + 1/x = N \tag{23}$$

as the N-th silver means of the first and second kind respectively and denote them as $SM_1(N)$ and $SM_2(N)$ (Kappraff, 2000b). When N = 1, $x = SM_1(1) = \tau$, the golden mean. Therefore, in Equation 22b, $\alpha = SM_1(N)$ or $\alpha = SM_2(N)$.

As a result of Equation 23, α satisfies one of the equations,

$$x^2 = Nx \pm 1 \tag{24}$$

Therefore, the sequence

$$1,\ \alpha^1,\ \alpha^2,\ \alpha^3,\ \alpha^4,\ \ldots \tag{25}$$

is a generalized Pell sequence (Kappraff, 2000b) and satisfies the recursion,

$$\alpha^n = N\alpha^{n-1} \pm \alpha^{n-2} \tag{26}$$

as does the sequence, $\{c_k\}$ where,

$$c_k = Nc_{k-1} \pm c_{k-2} \tag{27a}$$

where $\frac{c_{n+1}}{c_n}$ approaches α in the limit. We also find that when b = 2, Equation 20b has the special solution: $g_n = k$, i.e.,

$$c_n^2 - c_{k-1}c_{k-2} = k \quad \text{for all n} \tag{27b}$$

This means that if k = 0, the sequence $\{c_n\}$ is a geometric sequence. Otherwise it is an approximate geometric sequence.

We consider eight examples:

Example 1: a = 3, b = 2, N=1. $\alpha = SM_1(1) = \tau$ and $\beta = 1/\tau$, row and column elements are generated by $(2+3x)^n$, row and column sum = 5^n, Sequence 20 yields {0,1,3,8,21,…}, even indexed Fibonacci terms with ratio of successive terms approaching τ^2. In Equation 27b, we find that k = 1. The golden mean has found many applications. LeCorbusier made it the basis of his Modulor system of architectural design (Kappraff, 2000c).

Example 2: a = 6 , b = 2, N = 2. Replacing this into Equation 11 yields $\alpha = SM_1(2) = 1+\sqrt{2}$,

$\beta = \dfrac{1}{1+\sqrt{2}}$, row and column elements are generated by $(2+6x)^n$, row and column sum = 8^n

Sequence 20a yields: {0,1,6,35,204,…}, approaching β^2. In Equation 27b, we find that k = 1. The proportion, $1+\sqrt{2}$ is commonly know as the silver mean and was the basis of the system of proportions used in the Roman empire (Kappraff, 2000c).

Example 3: a = 5, b = 4, $\alpha = 2$, $\beta = 1$, row and column elements generated by $(4 + 5x)^n$, row and column sum = 9^n . Sequence 20, i.e., $c_n = 5c_{n-1} - 4c_{n-2}$, yields : {0,1,5,21,85,341,1365,…} = $\{c_n = \dfrac{4^n - 1}{3}\}$ as the ratio of successive terms tends to 4.

Example 4: a = 5, b = 3, $\alpha = 3/\sqrt{2}$, $\beta = 1/\sqrt{2}$, row and column elements generated by $(5+3x)^n$, row and column sum = 8^n. The generalized Nicomachus Triangle in Table 4 is generated from $\{3^{n-k} \times 5^k\}(0 \le k \le n)$.

Table 4. Generalized Nicomachus Triangle Generated by (3,5)

1				
3	5			
9	15	25		
27	45	75	125	
81	135	225	375	625

Each column of this Triangle represents a sequence of musical fifths and recreates the ancient Pythagorean scale, whereas any three successive columns generates the tones of the ancient Just scale (Kappraff, 2000), (McClain, 1976).

Example 5: a = 4, b = 3, $\alpha = \sqrt{\frac{4+\sqrt{7}}{2}}$, $\beta = \sqrt{\frac{4-\sqrt{7}}{2}}$, row and column elements generated by $(4+3x)^n$, row and column sum = 7^n.

Example 6: a = 7, b = 2, N = 3, $\alpha = SM_2(3) = \tau^2$, $\beta = 1/\tau^2$, row and column elements generated by $(2+7x)^n$, row and column sum = 9^n

Example 7: a = 1, b = 1, $\alpha = 1/\sqrt{2}$, $\beta = 1/\sqrt{2}$. Row and column elements are generated by $(1+x)^n$.

All elements of the generalized Nicomachus Triangle (see Table 3) are ones but taking into account multiplicity yields Pascal's Triangle (see Table 2) whose (n,k)-th element is equal to $\frac{n!}{k!(n-k)!}$.

The ratio of successive terms in Sequence 20, i.e., $c_1 = 0, c_2 = 1$ and $c_n = c_{n-1} - \frac{1}{4}c_{n-2}$, yields: 1, ¾, 2/3, 5/8, 3/5, 7/12, 4/7, … approaching the value of ½. These ratios are the fundamental, musical fourth, fifth, minor sixth, major sixth of the ancient Just scale, and two approximations to the major and minor sevenths, all approaching the octave value of ½. If the modulus M of a pair of successive approximating fractions a/b and c/d is defined as

M = (ad – bc) then all moduli of the approximating sequence have the value 1, e.g., (1x4 – 1x3) = 1, (3x3 – 4x2) = 1, etc. As a result, the approximating fractions appear as elements of successive rows of the Farey Table to the right of ½ (Kappraff, 2000b).

Example 8: a = 1, b = 2, N = i, $\alpha = SM_1(i) = e^{i\pi/6}$, $\beta = e^{-i\pi/6}$, row and column terms are generated by $(2+x)^n$, row and column sum = 3^n. The generalized Nicomachus Triangle yields,

Table 5. Generalized Nicomachus Triangle Generated by (1,2)

1				
2	1			
4	2	1		
8	4	2	1	
16	8	4	2	1

Multiplying the elements of Table 5 by the elements of Pascal's Triangle to account for multiplicity yields the square of Pascal's Triangle, a triangle whose (i,j)-th entry is (i,j)x2^{i-j} where (i,j) is the element of the i-th row and j-th column of Pascal's Triangle.

Table 6. Square of Pascal's Triangle

1				
2	1			
4	4	1		
8	12	6	1	
16	32	24	8	1

The rows of Table 6 give the number of vertices, edges, faces, cells, etc of hypercubes, of increasing dimension, e.g., H_0 (point) V = 1; H_1 (line segment) V = 2, E = 1; H_2 (square) V = 4, E = 4, F=1; H_3 (cube) V = 8, E = 12, F = 6, C = 1; H_4 (tesseract) V = 16, E = 32, F = 24, C = 8, hypercube = 1. Sequence 20 generates the sequence: 0,1,1,0,-1,-1,0,1,1,…, and the ratio of successive terms should approach $\alpha^2 = 1$. In fact, the ratio of a subsequence approaches 1 identically as it should. In Eq. 27b, we find once again that k = 1.

If a = 2, b = 1, then the generalized Nicomachus Triangle is identical to the one for a = 1, b = 2 but the columns are in reverse order. On the other hand $\alpha = \sqrt{\frac{2+\sqrt{3}}{2}}$ and $\beta = \sqrt{\frac{2-\sqrt{3}}{2}}$.

4. PYTHAGOREAN TRIPLES AHD THE SQUARE OF A 2x2 BISYMMETRIC MATRIX

For x,y real numbers with x>y, and

$$\mathrm{M} = \begin{bmatrix} x & y \\ y & x \end{bmatrix} \tag{28}$$

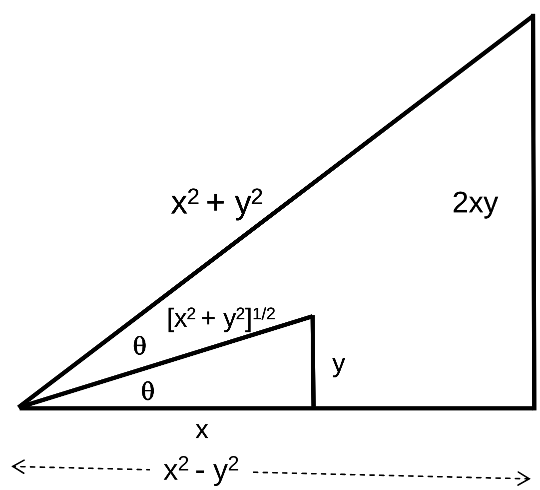
The square of this matrix is,

$$\mathrm{M}^2 = \begin{bmatrix} x & y \\ y & x \end{bmatrix}^2 = \begin{bmatrix} x^2+y^2 & 2xy \\ 2xy & x^2+y^2 \end{bmatrix} . \tag{29}$$

Notice that M^2 has values that are the hypotenuse and altitude of a right triangle whose base is $x^2 - y^2$. As a result, if x and y are integers, then $\{ x^2 + y^2, 2xy, x^2 - y^2 \}$ is a Pythagorean triple, i.e., three integer sides of a right triangle.

Compare this with the complex number x + iy and its square, $(x+iy)^2 = x^2 - y^2 + 2ixy$. Here the argument of x+iy is doubled while its modulus is squared, i.e., if $\tan\theta = y/x$ then $\tan 2\theta = \dfrac{2xy}{x^2-y^2}$ while the modulus squares from $\sqrt{x^2+y^2}$ to x^2+y^2 as shown in Figure 3. The hypotenuse of this triangle is x^2+y^2 so that it is identical with the triangle in Figure 3.

Figure 3. Pythagorean triples



We now identify $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$ with the ordered pair (x,y) or equivalently with the complex number x + iy so that $\tan\theta = y/x$ =c. It can be shown that the ordered pair, (c,1), corresponds to a triangle with the radius r of the inscribed circle, and area A given by,

$$r = c-1\,,\quad A = (c-1)(c)(c+1)$$

It follows that the radius r of the inscribed circle and the area of triangle (a,b) is,

$$r = b^2(a/b-1) = ab - b^2 \quad \text{and} \quad A = b^4(a/b-1)(a/b)(a/b+1) = ab(a^2-b^2) \qquad \text{(30a,b)}$$

It also follows from Equations 30 that,

$$r = \frac{area}{semiperimeter}$$

where this equation holds for triangles that are not right triangles.

5. SQUARE ROOT OF A 2X2 BISYMMETRIC MATRIX

It follows from Equation 29 that,

$$\begin{bmatrix} x^2+y^2 & 2xy \\ 2xy & x^2+y^2 \end{bmatrix}^{1/2} = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$$

We now pose the problem to find x and y such that,

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}^{1/2} = \begin{bmatrix} x & y \\ y & x \end{bmatrix}\,. \qquad (31)$$

where a is the hypotenuse and b is the altitude of a right triangle with vertex angle θ whose base is $\sqrt{a^2-b^2}$, $\tan\theta = b/\sqrt{a^2-b^2}$ and $\tan\theta/2 = y/x$. As a result, using standard trigonometric identities,

$$\tan\theta/2 = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \quad \text{where} \quad \cos\theta = \frac{\sqrt{a^2-b^2}}{a}$$

After some algebra,

$$\tan\theta/2 = \frac{a-\sqrt{a^2-b^2}}{b}$$

which implies that,

$$x = kb \text{ and } y = k(a-\sqrt{a^2-b^2}\ ). \tag{32}$$

But, since the hypotenuse of the right triangle with vertex $\theta/2$ is $\sqrt{a}$,

$$\sqrt{x^2+y^2} = \sqrt{a} \tag{33}$$

Replacing Equations 32 into 33 and solving for k it follows after some algebra that,

$$x = \frac{b}{2y} \quad \text{and} \quad y = \sqrt{\frac{a-\sqrt{a^2-b^2}}{2}}$$

which agrees with Equations 11a and b.

6. PYTHAGOREAN TRIPLES AND POWERS OF 2X2 BISYMMETRIC MATRICES

For a,b natural numbers with a>b, even powers of an arbitrary 2x2 bisymmetric matrix,

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}^{2n}$$

results in a sequence of pairs of whole numbers that are hypotenuse and side of Pythagorean triples for all values of n. The third side will be powers of a^2-b^2. If $\sqrt{a^2-b^2} = c$ for c a natural number, i.e., if {a,b,c} is a Pythagorean triple, then all powers of the bisymmetric matrix results in hypotenuse and side of Pythagorean triples with the third side being powers of c. It should be noted that the first number of these Pythagorean triples, a, represents the length of the hypotenuse unlike the first number of (a,b) which was the length of a side.

Example 1: (3,2)

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}^2 = \begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix}$$ Therefore the Pythagorean triple is {13,12,5}

$r = 3\times 2 - 2^2 = 2$ and $A = (3\times 2)(3^2 - 2^2) = 30$

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}^4 = \begin{bmatrix} 313 & 312 \\ 312 & 313 \end{bmatrix}$$ with {313,312,25}

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}^6 = \begin{bmatrix} 313 & 312 \\ 312 & 313 \end{bmatrix} \begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix} = \begin{bmatrix} 7813 & 7812 \\ 7812 & 7813 \end{bmatrix}$$ with {7813,7812,125}

Example 2: a=5, b=4 where {5,4,3} is a triple.

$$\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}^2 = \begin{bmatrix} 41 & 40 \\ 40 & 41 \end{bmatrix}$$ with {41,40,9}

where $r = 5\times 4 - 4^2 = 5$ and $A = (5\times 4)(5^2 - 4^2) = 180$

$$\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}^3 = \begin{bmatrix} 41 & 40 \\ 40 & 41 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 365 & 364 \\ 364 & 365 \end{bmatrix}$$ with {365,364,27}

7. CONCLUSION

Petoukhov's RNA matrices have led to a generalization of the golden mean, generalized Fibonacci sequences, generalized Nicomachus Triangles, and to an algorithm for generating Pythagorean triples. Pascal's Triangle plays an important role. Although Petoukov's matrices reproduce the sequences of musical fifths found in the rows of the Nicomachus Triangle, there is no obvious connection between the genomic matrices and the musical scale.

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APPENDIX A

Petoukhov's Matrix Genetics :

We have demonstrated the symmetry that is uncovered when the ensembles of genetic multiplets and other genetic elements are represented by the Petoukhov matrices, and we have shown how this symmetry results in analyzing certain patterns in the evolution of all of the known classes of genetic code (These classes are shown on the NCBI site http://www.ncbi.nlm.nih.gov/Taxonomy/Utils/wprintgc.cgi). We have also shown that generalizations of these matrices reveal sturdy mathematical structures bringing together generalizations of the golden mean, Fibonacci sequences, new characterizations of Pythagorean triples, and mathematical structures that generalize the musical scale of Pythagoras.

This work calls attention to the broader results that Petoukhov has expressed through his papers and books (Petoukhov, 2008 a,b). Petoukhov states that one of the most important tasks of science is to find a way of creating order in the study of genetic coding. The work of Petoukhov describes the utility of matrix methods to represent and to analyze hierarchical systems of genetic coding for mathematical classification and modeling of natural forms. This work demonstrates applications to the ordering of genetic code of special algebras, and other well-known mathematical structures such as Hadamard matrices, double numbers, transformations of hyperbolic turns, the golden section, the Pythagorean musical scale, etc. The results of his analysis suggest that living substances have their own forms of creating order and that these forms are connected with special algebras which are new to biomathematics. These algebras are related to special multidimensional geometries. They permit the development of new models in the fields of molecular genetics, bioinformatics and mathematical biology in general. A discovery of these algebras leads to the construction of biological theories on the basis of a language of biological algebras. Petoukhov's results suggest that many difficulties in the mathematizing of biology may be due to inappropriate numerical systems (algebras) which are utilized to describe biological structures. These difficulties can be compared to problem which Hamilton faced when he tried for many years to find a description of properties of 3D space by means of algebras of 3-dimensional numbers until he realized that these required the new four dimensional algebra of quaternions.

APPENDIX B

Theorem : $P_2^2 = M_2$ where,

$$M_2 = \begin{bmatrix} a^2 & ab & b^2 & ab \\ ab & a^2 & ab & b^2 \\ b^2 & ab & a^2 & ab \\ ab & b^2 & ab & a^2 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} \alpha^2 & \alpha\beta & \beta^2 & \alpha\beta \\ \alpha\beta & \alpha^2 & \alpha\beta & \beta^2 \\ \beta^2 & \alpha\beta & \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 & \alpha\beta & \alpha^2 \end{bmatrix}$$

.

Proof:

Let $P_1 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, $Q_1 = \begin{bmatrix} \beta & \alpha \\ \alpha & \beta \end{bmatrix}$., $M_1 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $N_1 = \begin{bmatrix} b & a \\ a & b \end{bmatrix}$

P_2 and M_2 can then be rewritten,

$$P_2 = \begin{bmatrix} \alpha P_1 & \beta Q_1 \\ \beta Q_1 & \alpha P_1 \end{bmatrix} \text{ and } M_2 = \begin{bmatrix} aM_1 & bN_1 \\ bN_1 & aM_1 \end{bmatrix}.$$

Using Eq. 12a and b, $a = \alpha^2 + \beta^2$, $b = 2\alpha\beta$, and it follows immediately that,

$$P_2^2 = M_2$$

It follows by induction that $P_n^2 = M_n$ where,

$$P_n = \begin{bmatrix} \alpha P_{n-1} & \beta Q_{n-1} \\ \beta Q_{n-1} & \alpha P_{n-1} \end{bmatrix} \text{ and } M_n = \begin{bmatrix} aM_{n-1} & bN_{n-1} \\ bN_{n-1} & aM_{n-1} \end{bmatrix}.$$